Exercises from Leinster

Michael Weiss

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In 2017–2018 I participated in a meetup grooup (Cambridge Advanced Math Studies Group) that went through Leinster's book *Basic Category Theory* [6]. This file is the end result of that.

I wish to thank all my fellow participants: Ken Halpern, Andras Pap, Matt Rumizen, Rob Sides, and Elliot Yu (apologies to anyone whose name has escaped my memory). However, any errors are mine alone.

1 Functors and Natural Transformations

1.1 1.2.24, p.26

There is no such functor Z. We let $Z(G)$ be the center of G for all G. Let $T \subseteq S_3$ be a transposition subgroup on the symmetric group on three letters, say $T = \{1, (ab)\}$. Let C_3 be the cyclic group of order three, which is the unique normal subgroup of S_3 (except for S_3 and 1). Since $S_3/C_3 \cong T$, and the transposition $(ab) \notin C_3$, there is an epimorphism $\varphi : S_3 \to T$ such that

$$T \overset{\iota}{\hookrightarrow} S_3 \overset{\varphi}{\to} T$$

gives the identity: $\varphi\iota = 1_T$. (Here, ι is inclusion.) So if there were a functor Z, we would have

$$Z(\varphi\iota) = Z(\varphi)Z(\iota) = 1_T$$

But $Z(S_3) = 1$, so $Z(\iota)$ must be the trivial homomorphism, and $Z(\varphi)Z(\iota) = 1_T$ is impossible.

1.2 1.2.29, p.27

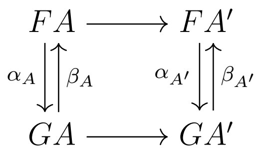
(a) Let $\mathcal{A}$ be a partially ordered set regarded as a category. If $\mathcal{B}$ is a subcategory of $\mathcal{A}$, then $b \leq_{\mathcal{B}} b' \Rightarrow b \leq_{\mathcal{A}} b'$—this condition is necessary and sufficient. (Informally: we erase some of the elements of $\mathcal{A}$ and some of the directed edges, to obtain $\mathcal{B}$.) For a full subcategory, $b \leq_{\mathcal{B}} b' \Leftrightarrow b \leq_{\mathcal{A}} b'$, so $\mathcal{B} = \mathcal{A}|\mathrm{Obj}(\mathcal{B})$.

(b) Subcategories of a group are the monoids contained in the group (with the same operation, of course). The only full subcategory of a group is the group itself.

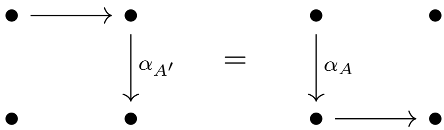
1.3 1.3.26, p.38

One direction is trivial: if α is a natural isomorphism with β its inverse, then $\alpha_A\beta_A = (\alpha\beta)_A = 1_{GA}$, $\beta_A\alpha_A = (\beta\alpha)_A = 1_{FA}$.

In the other direction, suppose each α_A has an inverse β_A (necessarily unique). We need to show that β is a natural transformation. The naturality square for α, with the β's indicated:



We symbolically indicate the proof of β's naturality with these diagrams:



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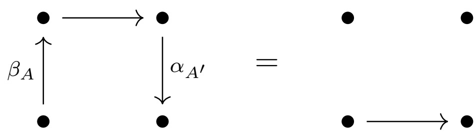

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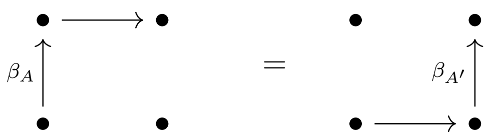


1.4 1.3.28, p.39

(a) $(a, f) \mapsto f(a)$.

(b) Using lambda notation: $a \mapsto \lambda f(f(a))$. In other words, $a \mapsto (f \mapsto f(a))$, where $f \mapsto f(a)$ is the function mapping $f \in B^A$ to $f(a) \in B$. So $(f \mapsto f(a)) \in B^{B^A}$, and $a \mapsto (f \mapsto f(a))$ maps $A \to B^{B^A}$.

1.5 1.3.29, p.39

$F^A : B \mapsto F(A, B)$, and if $g : B \to B'$, then $F(A, B) \xrightarrow{F(1_A, g)} F(A, B')$.

$F_B : A \mapsto F(A, B)$, and if $f : A \to A'$, then $F(A, B) \xrightarrow{F(f, 1_B)} F(A', B)$.

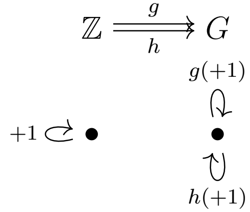
So:

$$\begin{array}{ccc}
F(A, B) & \xrightarrow{\alpha_{A,B}} & G(A, B) \\
{\scriptstyle F(f, 1_B)}\downarrow & & \downarrow{\scriptstyle G(f, 1_B)} \\
F(A', B) & \xrightarrow{\alpha_{A',B}} & G(A', B) \\
{\scriptstyle F(1_{A'}, g)}\downarrow & & \downarrow{\scriptstyle G(1_{A'}, g)} \\
F(A', B') & \xrightarrow{\alpha_{A',B'}} & G(A', B')
\end{array}$$

The top and bottom squares commute, so the whole square commutes.

1.6 1.3.30, p.39

The relationship is conjugacy. Suppose $f, g : \mathbb{Z} \to G$ are functors, as shown below; here, '+1' is the element $1 \in \mathbb{Z}$ (i.e., "add 1") and not the identity functor $1_{\mathbb{Z}}$ (which equals $0 \in \mathbb{Z}$).



As Leinster suggests, we write just g and h instead of $g(+1)$ and $h(+1)$ and think of g, h as elements of the group G. A natural isomorphism is a family of morphisms $\{\alpha_A\}$ indexed by the objects of G, but there's only one such object (which we denote by $\bullet$), so $\alpha : g \Rightarrow h$ amounts to a commutative diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{\alpha_\bullet} & \bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \bullet & \xrightarrow{\alpha_\bullet} & \bullet \end{array}$$

i.e., $h\alpha_\bullet = \alpha_\bullet g$, i.e., $g = \alpha_\bullet^{-1} h \alpha_\bullet$. So g is naturally isomorphic to h iff g is conjugate to h.

1.7 1.3.31, p.39

(a) For any $A, B \in \mathcal{B}$ and bijection $\varphi : A \to B$, we define:

$$\mathrm{Sym}(A) = \{\alpha : A \leftrightarrow A\}$$
$$\mathrm{Sym}(A) \xrightarrow{\mathrm{Sym}(\varphi)} \mathrm{Sym}(B)$$
$$\alpha \mapsto \varphi\alpha\varphi^{-1}$$

That is, $\mathrm{Sym}(A)$ is the set of all bijections of A to itself, and $\mathrm{Sym}(\varphi)$ maps $\alpha : A \leftrightarrow A$ to $\beta : B \leftrightarrow B$ via conjugation by φ.

Define

$$\mathrm{Ord}(A) = \{\leq_A : \leq_A \text{ totally orders } A\}$$
$$\mathrm{Ord}(A) \xrightarrow{\mathrm{Ord}(\varphi)} \mathrm{Ord}(B)$$
$$\leq_A \;\mapsto\; \leq_B$$
$$b \leq_B b' \Leftrightarrow \varphi^{-1}(b) \leq_A \varphi^{-1}(b')$$

That is, $\mathrm{Ord}(A)$ is the set of all linear orderings of A, and $\mathrm{Ord}(\varphi)$ uses φ to pull the ordering of A over to B. (Visually, we arrange the elements of A in a line, and then replace each element of A with the corresponding element of B.)

(b) We have

$$1_A \mapsto \varphi\, 1_A\, \varphi^{-1} = \varphi\varphi^{-1} = 1_B$$

so identities are preserved. If we had $\alpha : \mathrm{Sym} \Rightarrow \mathrm{Ord}$, then we would have, for the special case where $\varphi : A \leftrightarrow A$:

$$\begin{array}{ccc} \mathrm{Sym}(A) & \xrightarrow{\alpha_A} & \mathrm{Ord}(A) \\ {\scriptstyle \mathrm{Sym}(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ord}(\varphi)} \\ \mathrm{Sym}(A) & \xrightarrow[\alpha_A]{} & \mathrm{Ord}(A) \end{array}$$

and so

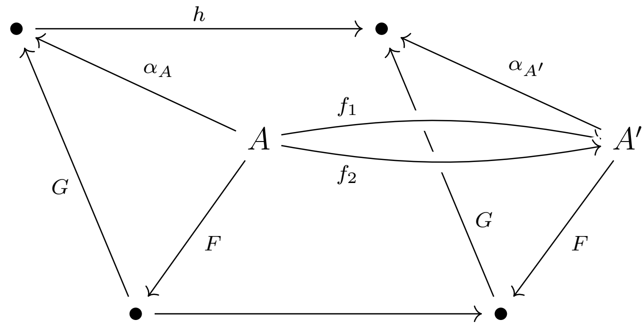
$$\begin{array}{ccc} 1_A & \overset{\alpha_A}{\longmapsto} & \alpha_A(1_A) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Ord}(\varphi)} \\ 1_A & \overset{\alpha_A}{\longmapsto} & \alpha_A(1_A) \end{array}$$

but if φ is not the identity on A, then $\mathrm{Ord}(\varphi)$ changes every total order on A.

(c) Both $\mathrm{Sym}(X)$ and $\mathrm{Ord}(X)$ have $n!$ elements if $|X| = n$.

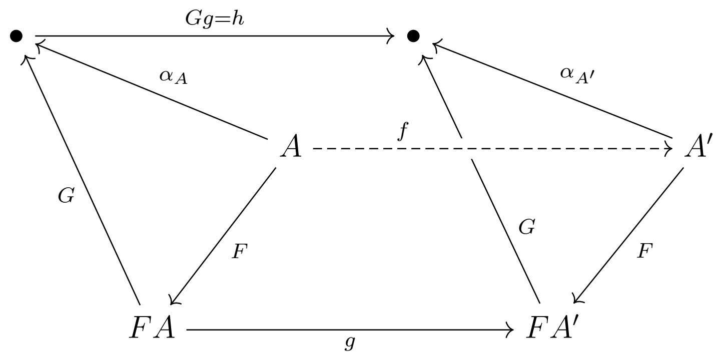
1.8 1.3.32, p.40

(a) Assume $F : \mathcal{A} \to \mathcal{B}$ is an equivalence, with $G : \mathcal{B} \to \mathcal{A}$ and $\alpha : 1_{\mathcal{A}} \Leftrightarrow GF$. First we prove faithfulness, so let $f_1, f_2 : A \to A'$ be morphisms, and let $F(f_1) = F(f_2)$. Then $GF(f_1) = GF(f_2) = h$ (say), and we have this situation:



So $f_1 = \alpha_{A'}^{-1} h \alpha_A = f_2$.

Next, fullness. Let $g : FA \to FA'$; we need to show there is an f with $Ff = g$, like so:

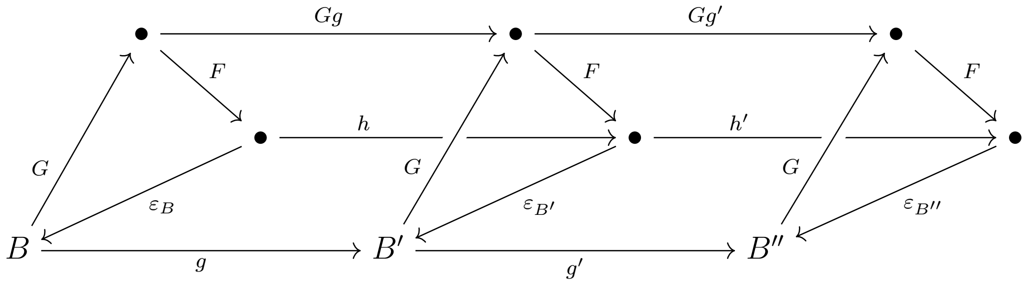


Set $h = Gg$ and set $f = \alpha_{A'}^{-1} h \alpha_A$. Now, $GFf = \alpha_{A'} f \alpha_A^{-1} = h$ because $\alpha : 1_{\mathcal{A}} \Leftrightarrow GF$. Since $G(Ff) = Gg$ and G is faithful, $Ff = g$, as desired.

Finally, essential surjectivity. But $B \cong F(GB)$ because $FG \Leftrightarrow 1_{\mathcal{B}}$.

(b) Suppose F is full, faithful, and essentially surjective. Since F is essentially surjective, for any $B \in \mathcal{B}$ we can choose an $A \in \mathcal{A}$ with $FA \cong B$; for each B, set $GB = A$ and choose an isomorphism $\varepsilon_B : FA \to B$.

Now we need to define Gg for each morphism $g : B \to B'$, defining it in a functorial fashion. This diagram gives the essentials:



We set $h = \varepsilon_{B'}^{-1} g \varepsilon_B$. Since F is full and faithful, F^{-1} is well-defined. So set $Gg = F^{-1}h$. We have the required naturality square with $g\varepsilon_B = \varepsilon_{B'}(FGg)$, showing that $\varepsilon : FG \Leftrightarrow 1_{\mathcal{B}}$. To see that G is a functor, look at this computation:

$$FG(g')FG(g) = \varepsilon_{B''}^{-1} g' \varepsilon_{B'} \varepsilon_{B'}^{-1} g \varepsilon_B = FG(g'g)$$

1.9 1.3.33, p.40

Composition in **Mat** is matrix multiplication. Define a functor $F : \mathbf{Mat} \to \mathbf{FDVect}$ by $F(n) = k^n$ (recall that k is the field), with $F(A)$ the linear map determined by the matrix A. This is clearly full, faithful, and essentially surjective. F is not canonical; we could, for example, choose an non-standard basis for each k^n (say the standard basis in reverse order) and then declare that A determines the linear map with respect to this basis.

1.10 1.3.34, p.40

Suppose we have $\mathcal{A} \underset{J}{\overset{F}{\rightleftarrows}} \mathcal{B} \underset{H}{\overset{G}{\rightleftarrows}} \mathcal{C}$, with $1_{\mathcal{A}} \Leftrightarrow JF$, $FJ \Leftrightarrow 1_{\mathcal{B}} \Leftrightarrow HG$, $GH \Leftrightarrow 1_{\mathcal{C}}$.

First proof: we use Prop. 1.3.18. All we need to show, then, is that GF is full, faithful, and essentially surjective on objects. This follows almost immediately from the fact that these three properties hold for F and G. (Essential surjectivity takes a moment of thought, fullness and faithfulness not even that.)

Second proof: We have to show that $1_{\mathcal{A}} \Leftrightarrow (JH)(GF)$ and $(GF)(JH) \Leftrightarrow 1_{\mathcal{C}}$. We compose natural isomorphisms:

$$\begin{aligned} 1_{\mathcal{B}} &\Leftrightarrow HG \\ F &\Leftrightarrow HGF \\ 1_{\mathcal{A}} &\Leftrightarrow JF \Leftrightarrow JHGF \end{aligned}$$

Here we first use horizontal composition (Leinster, p.37) followed by vertical composition (Leinster, p.30 and p.36) to get $1_{\mathcal{A}} \Leftrightarrow JHGF$. A similar argument shows that $GFJH \Leftrightarrow 1_{\mathcal{C}}$.

1.11 Monics and Epics

Monics and epics show up scattered throughout the exercises; this section collects some facts about them for ease of reference. Leinster introduces these concepts on pp.123, 134–136. See also §2.10, §2.11, and §5.12.

Here are the definitions:

- We say g is **monic** iff for all x, y, $gx = gy \Rightarrow x = y$. We say g is **split monic** iff g has a left inverse, say $hg = 1$.
- We say g is **epic** iff for all x, y, $xg = yg \Rightarrow x = y$. We say g is **split epic** iff g has a right inverse, say $gh = 1$.

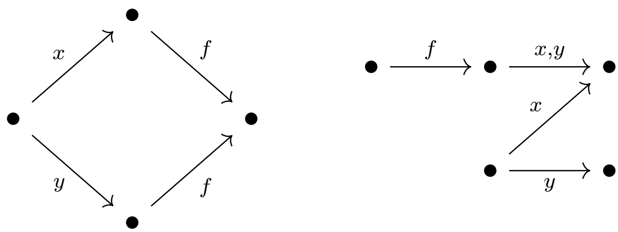
So monic means we can cancel on the left, epic on the right.

Some terminology, borrowed from topology: split monics are also called **sections**, and split epics **retracts**, especially in the case where the right inverse is an inclusion of sets.

Some easy facts:

1. Split monic $\Rightarrow$ monic, split epic $\Rightarrow$ epic. Proof: obvious.
2. Split monic plus epic $\Leftrightarrow$ isomorphism. Proof: Say g is both split monic and epic. So $hg = 1$ for some h. Then $ghg = g1 = 1g$, so cancelling on the right, $gh = 1$. The converse is obvious.
3. Split epic plus monic $\Leftrightarrow$ isomorphism. Proof: duality.
4. Composition of monics is monic, compositions of epic is epic. Proof: for monics f and g, $fgx = fgy \Rightarrow gx = gy \Rightarrow x = y$. Dually for epics.
5. Composition of split monics is split monic, likewise split epics. Proof: if f' and g' are left inverses to f and g, then $g'f'$ is a left inverse to fg. Ditto for right inverses. (Note that neither left nor right inverses need be unique.)
6. A monic composition has a monic on the right. Proof: look at the contrapositive. If $x \neq y$ and $gx = gy$, then $fgx = fgy$, so g not monic implies fg not monic.
7. An epic composition has an epic on the left. Proof: duality.
8. Faithful functors "pull back" monics and epics. That is, if U is faithful and Uf is monic/epic, then f is monic/epic. Proof: if Uf is monic, then $fx = fy \Rightarrow (Uf)(Ux) = (Uf)(Uy) \Rightarrow Ux = Uy \Rightarrow x = y$. Likewise for epics.
9. Functors "push forward" split monics and split epics. That is, if f is split monic/epic, then Uf is split monic/epic for any functor U. Proof: $ef = 1 \Rightarrow (Ue)(Uf) = 1$, and likewise for epics.

Two pictures can help you tell your left from your right with monics and epics. Both come from **Set**.



On the left, we have a non-monic f, erasing the difference between x and y: $x \neq y$ but $fx = fy$. On the right, a non-epic f does the same on the other side: $x \neq y$ but $xf = yf$.

In **Set**, split monic implies injective, and injective iff monic; split epic iff surjective iff epic. Split monic is *almost* equivalent to injective, but there is a sole class of exceptions: empty functions to a nonempty set, i.e., $f : \varnothing \to A \neq \varnothing$. Such an f is injective but not split monic, since we can't have a function from a nonempty set to the empty set. (Proofs of these facts are in §5.15(b).)

A **concrete** category has a faithful functor to **Set**. Examples: **Monoid**, **Group**, **Top**. Key point: two morphisms (with the same source and target) have to differ set-theoretically in order to be different. In **Top**, for example, given $f, g : X \to Y$, if $f(x) = g(x)$ for all $x \in X$, then $f = g$—topology doesn't impose any further constraints.

Say $\mathcal{A}$ is a concrete category with faithful functor U to **Set**. We can draw the following conclusions:

(a) f split monic $\Rightarrow$ Uf injective $\Rightarrow$ f monic.

(b) f split epic $\Rightarrow$ Uf surjective $\Rightarrow$ f epic.

Proof: By (8) above, if Uf is monic then f is monic, but monic for **Set** is the same as injective. By (9), if f is split monic then Uf is split monic (for all U, not just faithful ones), and split monic implies injective in **Set**.

So Uf injective implies f monic for a concrete category. Surprisingly often the converse holds: all monics are injective in **Monoid**, **Group**, **Top**, and many other concrete categories. This is a special case of the general fact that functors with left adjoints preserve limits (see Theorem 6.3.1 (p.159) plus Lemma 5.1.32 (p.124)).

It's not hard to show directly that functors with left adjoints preserve monics (dually for epics). Suppose that $F \dashv U$ with $F : \mathcal{A} \to \mathcal{B}$, $U : \mathcal{B} \to \mathcal{A}$. Let $B, B' \in \mathcal{B}$, and $f : B \to B'$. Assume f is monic. We want to show that Uf is monic in $\mathcal{A}$, so we consider $A \in \mathcal{A}$ and $\bar{x}, \bar{y} : A \to UB$. (The bar map is a bijection between $\mathcal{A}(A, UB)$ and $\mathcal{B}(FA, B)$, so we lose no generality in writing arbitary elements of $\mathcal{A}(A, UB)$ as $\bar{x}$ and $\bar{y}$.) We now use the naturality formula for adjunctions, $\overline{fx} = (Uf)\bar{x}$, in the chain of implications

$$
\begin{aligned}
(Uf)\bar{x} = (Uf)\bar{y} &\Rightarrow \\
\overline{fx} = \overline{fy} &\Rightarrow \\
fx = fy &\Rightarrow \\
x = y &\Rightarrow \\
\bar{x} = \bar{y}
\end{aligned}
$$

Done. Recording this fact:

10. A functor with a left adjoint preserves monics, and a functor with a right adjoint preserves epics.

Now let $\mathcal{A}$ be **Set**. So if $U : \mathcal{B} \to$ **Set** is a functor with a left adjoint, then f monic implies Uf monic, i.e., Uf injective. Dually, if U has a right adjoint, then it preserves epics. Right adjoints to forgetful functors are less common, though not unheard of (see below). Hand-in-hand with this, we find epics that are not surjective. Example 5.2.19, p.134 (same as Exericse 5.2.23, p.135, §5.12) gives the examples $\mathbb{N} \subseteq \mathbb{Z}$ in **Monoid**, and $\mathbb{Z} \subseteq \mathbb{Q}$ in **Ring**. So in neither case can U have a right adjoint.

Another case of interest, in the interaction between epics and adjoints: the functor F : **Monoid** $\to$ **Group**, sending a monoid to its universal enveloping group (see §2.2). This has a right adjoint U, so F preserves epics. Now we've seen that $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an epic inclusion in **Monoid**, so $F\mathbb{N} \to F\mathbb{Z}$ must be an epic morphism in **Group**. But all epics are surjective in **Group** (see §2.11). It so happens that $F\mathbb{N} = \mathbb{Z}$ and $F\mathbb{Z} = \mathbb{Z}$ and F of the inclusion is the identity map, so all is well.

Along the same lines, item (8) (faithful functors pull back epics) tells us this: a monoid epimorphism whose source and target both happen to be groups is surjective. Proof: any such morphism is $Uf : UG \to UH$ where U : **Group** $\to$ **Monoid** is forgetful. U is faithful, so since Uf is epic, f is epic, and epics in **Group** are surjective.

Leinster defines regular monics and epics in Exercises 5.2.25 and 5.2.26 (pp.135,136). A monic is **regular** if it is an equalizer (Definition 5.1.11, p.112). Likewise, an epic is **regular** if it is a coequalizer (Definition 5.2.7, p.129). The most important facts about regular monics and epics:

11. Split monic $\Rightarrow$ regular monic $\Rightarrow$ monic.

12. Split epic $\Rightarrow$ regular epic $\Rightarrow$ epic.

13. Regular monic plus epic $\Leftrightarrow$ isomorphism. (This strengthens (2) in one direction.)

14. Regular epic plus monic $\Leftrightarrow$ isomorphism. (This strengthens (3) in one direction.)

15. In **Set**, split epic ⇔ regular epic ⇔ epic ⇔ surjective. Also, split monic ⇒ regular monic ⇔ monic ⇔ injective, and injective plus nonempty source ⇒ split monic.

See §5.14 for proofs of (11), §5.15 for (12), (13), and (14). §5.15(b) has the proofs of (15).

1.12 Monics and Epics in Top

We look at all three varieties of monics and epics in **Top**, and its full subcategories **Hausdorff** (Hausdorff spaces) and **CptHff** (compact Hausdorff spaces).

Some cross-references. §5.7(c): monics in **Top**. §5.11(b): a curious example of a regular epic. §5.13: quotient objects. §5.14(c): regular monics in **Top**. §5.15(c): an epic that is regular but not split. §5.16: regular monics in **FHaus**, and an epic in **Hausdorff** that is not pullback stable.

Summary of results:

- Monics in **Top**, or in **Hausdorff**, are homeomorphisms onto *subsets* of a space where the homeomorphic image has a finer topology than the subspace topology.
- Regular monics in **Top** are homeomorphisms onto subspaces.
- Regular monics in **Hausdorff** are homeomorphisms onto closed subspaces.
- Monics in **CptHff** are all regular, and are homeomorphisms onto closed subspaces.
- Epics in **Top** are surjective maps.

- Epics in **Hausdorff** are maps with dense images.
- Epics in **CptHff** are all regular, and are quotient maps.
- Regular epics in **Top**, or in **Hausdorff**, are quotient maps.
- Neither all regular monics nor all regular epics are split, in any of these three categories.

(In this list, "are" is used in the sense of "if and only if". For example, a map in **Top** is regular monic if and only if it is a homeomorphism onto a subspace. Below however "are" is used in the "if-then" sense.)

Some basic definitions/facts from point-set topology:

- If $\mathcal{S}$ and $\mathcal{T}$ are topologies on the same set, we say $\mathcal{S}$ is **coarser** than $\mathcal{T}$, and $\mathcal{T}$ is **finer** than $\mathcal{S}$, iff $\mathcal{S} \subseteq \mathcal{T}$.
- A **quotient map** $f : X \to Y$ is a surjective map such that V is open in Y iff $f^{-1}(V)$ is open in X. Note that the $\Rightarrow$ implication is just the definition of continuity. Also note that we can replace "open" with "closed" and have an equivalent definition.
- An **open map** sends open sets to open sets. A **closed map** sends closed sets to closed sets.
- It follows from the equation $f(f^{-1}(V)) = V$, true for any surjective f, that any surjective open or closed map is a quotient map.
- If $s, t : X \to Y$ are continuous and Y is Hausdorff, then the **solution set** $\{x \in X | s(x) = t(x)\}$ is closed.
- The continuous image of a compact set is compact.
- A compact subset of a Hausdorff space is closed.

- A closed subset of a compact space is compact.
- Putting the last three facts together, any map from a compact space to a Hausdorff space is closed. That's all maps in **CptHff**!
- Finally, the forgetful functor U : **Top** $\to$ **Set** is faithful, and has a left adjoint (the discretization functor) and a right adjoint (the indiscretization functor). Replacing **Top** with **Hausdorff**, we still have the left adjoint, but lose the right adjoint.

I don't know of any slick characterization of split monics or split epics in any of these three categories.

The characterizations on our list shower us with boatloads of contrast examples: monics and epics that are not regular, in both **Top** and **Hausdorff**. For a non-regular monic, consider any inclusion map $i : A \hookrightarrow X$, where the topology of A is (strictly) larger than the subspace topology inherited from X. For a non-regular epic in **Top**, consider any surjective map $f : X \to Y$ where the topology of Y is (strictly) smaller than the quotient topology induced by f. Example: $f : \mathbb{R} \to \mathbb{R}$, Uf the identity, with the first $\mathbb{R}$ given the discrete topology and second the standard topology, is both monic and epic in both **Top** and **Hausdorff**, but not regular monic or regular epic in either. In **Hausdorff**, another family of non-regular epics consists of non-surjections with a dense images, e.g., $i : \mathbb{Q} \hookrightarrow \mathbb{R}$ with both given the standard topologies.

Next on our list of contrast examples, a regular monic that is not split, in all three of our categories: the inclusion $i : \{0,1\} \hookrightarrow [0,1]$. If we had $f : [0,1] \to \{0,1\}$ with $fi = 1_{\{0,1\}}$, this would mean that $f^{-1}(0) \sqcup f^{-1}(1)$ was a partition of $[0,1]$ into two disjoint nonempty closed sets, contradicting the fact that $[0,1]$ is connected. (This argument can be rephrased to use the intermediate value theorem.) A more sophisticated variant: $i : S^n \hookrightarrow D^{n+1}$, the inclusion of the n-sphere in the closed $(n+1)$-disk. (The $\{0,1\} \hookrightarrow [0,1]$ example is the case $n = 0$.) The homology functor proves that no left inverse exists, for if $fi = 1_{S^n}$, then the composition $H_n(S^n) \xrightarrow{Hi} H_n(D^{n+1}) \xrightarrow{Hf} H_n(S^n)$ would have to be the identity, but for $n > 0$, this is just $\mathbb{Z} \to 0 \to \mathbb{Z}$.[1] Regularity follows from the characterization given in the list above.

For a regular epic that is not split, consider $f : [0, 2\pi] \to S^1$ via the identification of 0 and 2π, i.e., $x \mapsto e^{ix}$. This is regular in all three categories since it's a quotient map (closed since it's from a compact space to a Hausdorff space). An equation $fi = 1_{S^1}$ is impossible: apply the fundamental group functor π (I apologize for the unfortunate double meaning of the same greek letter) to get $\pi(S^1) \xrightarrow{\pi i} \pi([0, 2\pi]) \xrightarrow{\pi f} \pi(S^1)$, or $\mathbb{Z} \to 0 \to \mathbb{Z}$. (The homology functor H_1 could be used instead.) Another example: $\mathbb{R} \to S^1$, with $x \mapsto e^{ix}$ as before. This works for **Top** and **Hausdorff** but not **CptHff**. The map is open and hence a quotient map.

We turn to the list of characterizations. First, monics in all three categories. If $m : X \to Y$ is injective, then it's monic, since U pulls back monics in **Set** (item (8) in §1.11). Conversely, if m is not injective, then it's not monic: if $m(x_1) = m(x_2)$, let $A = \{0\}$ (there's only one possible topology), and and set $s(0) = x_1$, $t(0) = x_2$. Then $ms = mt$ but $s \neq t$. (For a more sophisticated proof in **Top** and **Hausdorff**, note that U has a left adjoint and so preserves monics.)

Since m is continuous, the inclusion $m(X) \hookrightarrow Y$ is also continuous, if we give $m(X)$ the topology making m a homeomorphism. So the topology making $m(X)$ homeomorphic to X is finer than the subspace topology on $m(X)$.

In **CptHff**, m is automatically a closed map and so the bijection $X \xrightarrow{m} m(X)$ is a homeomorphism, with $m(X)$ having the subspace topology. Also $m(X)$ must be closed in Y. So in **CptHff**, a monic is an imbedding of one space as a closed subspace of another.

[1] For $n = 0$, you get the identity as a composition $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. With a little work, you can show that this is impossible. Using the reduced homology functor avoids this niggle. Of course, the case $n = 0$ has already been treated.

Next, regular monics. Suppose $s, t : X \to Y$. Let $S = \{x \in X | s(x) = t(x)\}$ with the subspace topology. The inclusion $i : S \hookrightarrow X$ equalizes s and t: on the one hand, clearly $si = ti$. On the other hand, if $f : A \to X$ satisfies $sf = tf$, then clearly $f(A) \subseteq S$. Define $e : A \to S$ by $e(a) = f(a)$ for all $a \in A$. So $ie = f$, and clearly e is the unique solution to this equation.

We know equalizers are unique up to isomorphism, so any other equalizer of s and t is a homeomorphism of some space onto S.

So in all three categories, a regular monic is a homeomorphism onto a solution set $\{x \in X | s(x) = t(x)\}$ for some s and t. We now have to determine what subsets of X can be expressed in this way.

In **Top**, every subset S of X is a solution set: let $Y = \{0, 1\}$ with the indiscrete topology, let $s(x) = 1$ for all $x \in X$ and let $t = \chi_S$, the characteristic function of S. Both s and t are continuous because every function to an indiscrete space is continuous, and obviously the solution set for s and t is S. So regular monics in **Top** are precisely homeomorphisms onto subspaces.

In **Hausdorff**, all solution sets are closed, as we noted above. We now show that conversely, all closed subspaces of X are solution sets. Let $X_{\text{dbl}} = X \sqcup X$, i.e., the disjoint union of two copies of X. Glue the two copies along S to obtain Y. (A useful mental picture: two copies of a closed disk, glued along a line segment.) In detail, we let X_1 and X_2 be the two copies, and i_1, i_2 be the obvious maps from X to X_1 and X_2 respectively. Form the quotient space of X_{dbl} by identifying $i_1(x)$ with $i_2(x)$ for all $x \in S$. Let $q : X_{\text{dbl}} \to Y$ be the quotient map, and let $s = qi_1$, $t = qi_2$. (Mental picture: s takes the disk to the "upper layer", and t to the "lower layer", but s and t agree on the line segment.) Let us note that we have a continuous surjection $r : Y \to X$ where we send each $y \in Y$ to its "grandparent" $x \in X$, i.e., we identify $s(x)$ with $t(x)$ for *all* $x \in X$, getting an identical copy of X. (Mental picture: we flatten the two layers into one.) So $rs = rt = 1_X$.

It remains to show that Y is Hausdorff. Suppose $T = s(X) = t(X)$, the identified subset, and let $y, y' \in Y$. Let $x = r(y)$, $x' = r(y')$. Case 1: $x \neq x'$. So there are disjoint open sets $U \ni x$, $U' \ni x'$. Set $V = r^{-1}(U)$ and $V' = r^{-1}(U')$ and we have disjoint open sets around y and y'. Case 2: $x = x'$. The only way to have $y \neq y'$ is if they live in different layers, i.e., $x \notin S$ and $y = s(x)$ and $y' = t(x)$ (or vice versa). Since S is closed, $X_1 \setminus i_1(S)$ and $X_2 \setminus i_2(S)$ are both open in X_{dbl}, and clearly q sends them to disjoint open sets in Y, one containing y and the other containing y'. So Y is Hausdorff.

Conclusion: the regular monics in **Hausdorff** are precisely the homeomorphisms onto closed subspaces.

The same argument works for **CptHff**, since X_{dbl} and Y are compact whenever X is. But we observed above that this also characterizes the monics in **CptHff**, so in this category all monics are regular.

Very soon we will reuse part of these arguments, so let's make it into a lemma:

> **Lemma:** in **Top**, any subset $S \subseteq X$ can be expressed as $S = \{x \in X | s(x) = t(x)\}$, where $s, t : X \to Y$ are continuous. In both **Hausdorff** and **CptHff**, any *closed* subset $S \subseteq X$ can be expressed this way. (Proof: see above.)

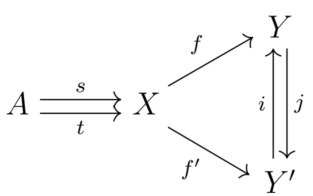
We turn to epics in **Top**. Any surjective map is epic, since U pulls back epics, and surjectives in **Set** are epic (items 8 and 15 in §1.11). On the other hand, U has a right adjoint $I : \textbf{Set} \to \textbf{Top}$, where IX is the indiscrete space on X. It follows that epics are surjective by item (10).

Here's a direct proof, in contrapositive form. Suppose $f : A \to X$ is not surjective. Let $f(A) = S$ and let $x_0 \in X \setminus S$. By the lemma, $S = \{x \in X | s(x) = t(x)\}$ with $s, t : X \to Y$. So $s(x_0) \neq t(x_0)$ and hence $s \neq t$, but $sf = tf$. So f is not epic.

Turning to **Hausdorff**, we have a similar argument but a different conclusion. By basic topology, if s and t agree on a dense subset of X, then they are equal (provided the target space is Hausdorff). It follows that if $f : A \to X$ has a dense image, then f is epic in **Hausdorff** because $sf = tf$ means that s and t agree on $f(A)$. On the other hand, if f is epic in **Hausdorff**, then it has a dense image. For if not, then $\overline{f(A)}$ is a proper subset of X. We let $S = \overline{f(A)}$, and appeal to the lemma. If $x_0 \in X \setminus S$, then $s(x_0) \neq t(x_0)$ and so $s \neq t$, but $sf = tf$, all as before. Conclusion: epics in **Hausdorff** are precisely the maps with dense images. (Leinster mentions half of this in Example 5.2.20, p.134.)

Finally, for **CptHff** the same argument works, but since *all* images in **CptHff** are closed, we're back to epic implying surjective. Now we noted in our list of basic topological facts that all maps in **CptHff** are closed, and that all surjective closed maps are quotient maps. So epics in **CptHff** are precisely the quotient maps.

Next, regular epics in **Top**. First, we show that regular epic implies quotient map. Suppose f is regular epic. Then f is epic and thus surjective. Suppose $f : X \to Y$ coequalizes $s, t : A \to X$. Let Y' have the same underlying set as Y but give it the quotient topology induced by f. Let $i : Y' \to Y$ be the identity set-theoretically, i.e., $Ui = 1_{UY}$. Then i is continuous since the quotient topology is finer than the topology of Y. (In detail: V open in Y implies $f^{-1}(V)$ open in X implies V open in Y'.) We let $f' : X \to Y'$ be the same map as f, set-theoretically ($Uf = Uf'$), so since $fs = ft$, $f's = f't$. Since f is a coequalizer, there is a unique $j : Y \to Y'$ with $jf = f'$. Hence in the diagram



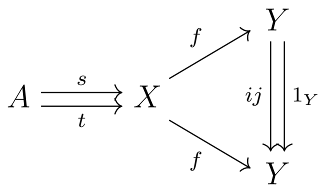
we have $jf = f'$, $if' = f$. So $ijf = f$ and $jif' = f'$, which means that

ij and ji are identities when restricted to the images of f and f'. But those images are Y and Y' since f and f' are surjective, so Y and Y' are isomorphic (homeomorphic), and Y has the quotient topology, i.e., f is a quotient map.

On the other hand, let's start with a quotient map $f : X \to Y$. This defines an equivalence relation $\sim$ on X: $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Let $A \subseteq X \times X$ be $\sim$ regarded as a set of ordered pairs, i.e., $A = \{(x_1, x_2) | f(x_1) = f(x_2)\}$. We give $X \times X$ the product topology and A the subspace topology. We let p_1 and p_2 be the projection maps restricted to A. So for all $a \in A$, $p_1(a) \sim p_2(a)$ and hence $fp_1 = fp_2$. Suppose $f' : X \to Y'$ also satisfies $f'p_1 = f'p_2$. We conclude that if $x_1 \sim x_2$, then $(x_1, x_2) = a \in A$ and so $f'(p_1(a)) = f'(p_2(a))$, i.e., $f'(x_1) = f'(x_2)$. That is to say, $f(x_1) = f(x_2)$ implies $f'(x_1) = f'(x_2)$. This, and the surjectivity of f, allow us to define (set-theoretically) a function $j : Y \to Y'$ with $jf = f'$: $j(y) = f'(f^{-1}(y))$. Moreover j is uniquely determined by the condition $jf = f'$. Finally, j is continuous because V open in Y' implies $f'^{-1}(V) = f^{-1}(j^{-1}(V))$ open in X implies $j^{-1}(V)$ open in X (as f is a quotient map).

How do we adapt these arguments to **Hausdorff**? In the $\Rightarrow$ direction, we don't know up front that f is surjective, only that its image is dense (because f is epic). Let Y' be the image with the induced quotient topology. Let $i : Y' \to Y$ be set-theoretically the inclusion map ($Ui : UY' \hookrightarrow UY$), and let $f' : X \to Y'$ be "the same" as f except for codomain (i.e., $f'(x) = f(x)$ for all $x \in X$, so $if' = f$; f' is a quotient map). This quotient topology is finer than the subspace topology inherited from Y (use the identity $f^{-1}(V) = f'^{-1}(i^{-1}(V))$), implying that Y' is Hausdorff and that the map $i : Y' \to Y$ is continuous. The argument then proceeds as before until we reach the point where we said that ij and ji are the identities on the images of f and f'. Since f' is surjective, $ji = 1_{Y'}$. The image of f is dense, so ij agrees with 1_Y on a dense subset. But in **Hausdorff**, two continuous functions that agree on a dense subset are equal. So $ij = 1_Y$, and we conclude both that f is surjective and that it's a quotient map.

We can also give a more abstract proof that $ij = 1_Y$, not using that fact about Hausdorff spaces. Consider the diagram



The uniqueness property of coequalizers demands that $ij = 1_Y$. This alternate argument proves handy in §5.16.

In the $\Leftarrow$ direction, the argument goes through without modification. However, we can add a remark: the subset $A \subseteq X \times X$ is defined by the equation $f(p_1(a)) = f(p_2(a))$ (now letting p_1 and p_2 be the projections from $X \times X$ to X), and so A is closed.

Finally, a regular epic in **CptHff** is a quotient map by the exact same argument as for **Hausdorff**, and a quotient map is regular epic because A is closed and so compact. So the regular epics are exactly the quotient maps. We noted earlier that this also describes the epics in **CptHff**.

One caveat: suppose a set $A \subseteq X \times X$ represents an equivalence relation $\sim$ on X. For $X/{\sim}$ to be Hausdorff, it is necessary but *not sufficient* that A be closed. This has the following consequence. Say we have $s, t : X \to Y$ in **Hausdorff**. We want to construct the coequalizer of s and t. As a first step, we construct the smallest equivalence relation $\sim$ such that $s(x) \sim t(x)$ for all $x \in X$ (see Leinster, pp.129–130). Taking the quotient $X \to X/{\sim}$ can take us out of the category of Hausdorff spaces: $X/{\sim}$ might not be Hausdorff. To finish the job, we need the **Hausdorffification** functor H : **Top** $\to$ **Hausdorff**: HY is the "largest" Hausdorff space for which there is a quotient map $Y \to HY$ (largest in the sense of a universal property). The construction of the Hausdorffification functor (aka Hausdorfization, Hausdorffication) is decidedly nontrivial (see §A.4).

To wrap things up, three sufficient conditions for a surjective f to be a quotient map: (a) f is an open map. (b) f is a closed map. (c) f is split epic. We observed (a) and (b) in our list of basic topological facts; (c) follows from item (12) in §1.11 (split epic $\Rightarrow$ regular epic) because regular epic is equivalent to being a quotient map.

Here are some examples to show that these conditions are not necessary. Let $Y = \mathbb{R}$ and let X be the subset of $\mathbb{R}^2$ consisting of the union of the x-axis and the closed right half plane, i.e., $\{(x, y)|x \geq 0\}$. Let f be the projection onto the y-coordinate. Then f is split epic with the right inverse being the inclusion $Y \hookrightarrow X$. But f is neither open nor closed. For example, intersecting an open disk centered at the origin gives an open subset of X whose projection is not open. And one branch of a hyperbola, namely $\{(x, y)|xy = 1\}$, is a closed subset of X whose projection is not closed. Next, consider again either $\mathbb{R}$ or $[0, 2\pi]$ mapping onto the circle ($f(x) = e^{ix}$). Both are quotient maps (one is open, the other closed), but as we showed above, neither is split.

1.13 Monics and Epics in Group

Monics in **Group** are precisely the injective homomorphisms. Injective $\Rightarrow$ monic because the forgetful functor U : **Group** $\to$ **Set** is faithful, and faithful functors pull back monics, and in **Set** the monics are precisely the injectives (items (8) and (15) of §1.11). Monic $\Rightarrow$ injective because U has a left adjoint (the free group functor) and functors with left adjoints preserve monics (item (10)). There's also an easy direct proof: if $f : G \to Q$ is not injective, then $\ker f$ is not 0, so we have two different homomorphisms $s, t : \ker f \to G$ satisfying $fs = ft$, namely s the inclusion map $\ker f \hookrightarrow G$ and t the trivial map $t(\ker f) = 1$ (the identity of G): $\ker f \underset{t}{\overset{s}{\rightrightarrows}} G \xrightarrow{f} Q$.

Epics in **Group** are precisely the surjective homomorphisms. Surjective $\Rightarrow$ epic: basically the same as injective $\Rightarrow$ monic. (Faithful functors pull back epics, and in **Set** the epics are precisely the surjectives.)

The proof that epic $\Rightarrow$ surjective is not easy. Here is a proof rewritten from Linderholm [7]; that reference says, "The present proof closely resembles one due to S. Eilenberg and J. Moore. . . it seems to be well known to those who work with categories."

Suppose $f : A \to G$ is not surjective. We want $s, t : G \to Q$ with $sf = tf$ but $s \neq t$. Let $H = f(A)$. So we want $s|H$ and $t|H$ to be equal with $s \neq t$:

$$H \hookrightarrow G \overset{s}{\underset{t}{\rightrightarrows}} Q$$

Hence it's enough to show that given any proper subgroup H of a group G, there are two different homomorphisms from G that agree on H. Actually we will show something stronger: there are two homomomorphisms that agree on H and disagree on all elements outside of H. Let G/H be the set of all left cosets of H. Let S be G/H plus one more element, which we'll denote by Z. Let Σ_S be the group of all permutations of S. Our homomorphisms will go from G to Σ_S.

Multiplication on the left by elements of G permutes the left cosets of H; let's write $\lambda_g : (G/H) \to (G/H)$ for this permutation. We extend λ_g to S by stipulating that $\lambda_g(Z) = Z$. Obviously the map $g \mapsto \lambda_g$ is a homomorphism from G to Σ_S (adopting the usual convention of right-to-left composition). We let $\lambda : G \to \Sigma_S$ be this homomorphism.

Let s be the permutation of S that interchanges H and Z (both elements of S) and leaves everything else fixed. Let σ_g be the permutation $\sigma_g(X) = s(\lambda_g(s^{-1}(X)))$, i.e., we conjugate the permutation λ_g by the permutation s. That is, $\sigma_g = (\lambda_g)^s$, adopting a common notation for conjugation in a group. (Since $s^2 = 1$, we could have just written $s(\lambda_g(s(X)))$, but this won't matter.) It is evident that $\sigma : g \mapsto \sigma_g$ is a homomorphism from G to Σ_S: $hk \mapsto (\lambda_{hk})^s = (\lambda_h \lambda_k)^s = (\lambda_h)^s(\lambda_k)^s$.

Now suppose h is an element of H. Since λ_h leaves H and Z unmoved, and

s just switches them leaving all the other elements of S unmoved, we see that λ_h and s commute. So $(\lambda_h)^s = \lambda_h$, and $\sigma_h = \lambda_h$. Therefore λ and σ agree on H.

On the other hand, let $g \notin H$. Then λ_g sends H to $gH \neq H$. We now compare λ_g with σ_g, or what amounts to the same thing, $s\lambda_g$ with $\lambda_g s$. We have $s(\lambda_g(Z)) = s(Z) = H$, but $\lambda_g(s(Z)) = \lambda_g(H) = gH$. Alternately, compute $s(\lambda_g(H)) = s(gH) = gH$, but $\lambda_g(s(H)) = \lambda_g(Z) = Z$. So $s\lambda_g \neq \lambda_g s$ and $\sigma_g \neq \lambda_g$.

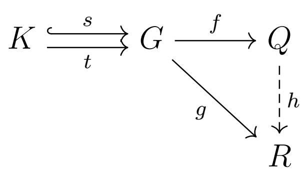
So $\lambda_g = \sigma_g$ precisely when g is in H. QED

Because our target group is finite whenever G is, this proof also works for **FinGroup**. Another common proof, using amalgamated free products, doesn't enjoy this feature.

In **Group**, all monics and all epics are regular. Leinster gives the key idea for epics in Example 5.1.14 (p.114). We repeat the argument. Suppose $f : G \to Q$ is epic. We want $s, t : K \to G$ such that f is the coequalizer of s and t. Let $K = \ker f$, let s be the inclusion map $K \hookrightarrow G$ and let t be the trivial map $t(K) = 1$. Clearly $fs = ft$. Suppose $g : G \to R$ with $gs = gt$. In other words, for every $x \in K$, $g(s(x)) = g(t(x))$, so $g(x) = g(1) = 1$, so $K = \ker f \subseteq \ker g$. This, plus the fact that f is surjective, implies that the function

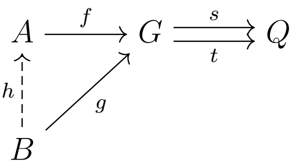
$$h : Q \to R$$
$$h : q \mapsto g(f^{-1}(q))$$

is well-defined. Here's the diagram:



It's routine to check that it's a homomorphism, that it satisfies $hf = g$, and that it's the unique solution to this equation. So f is a coequalizer.

For a monic $f : A \to G$, we want $s, t : G \to Q$ such that f equalizes s and t. In other words, s and t should agree on the image of f, and if that's also true for $g : B \to G$, then we want a unique $h : B \to A$ with $fh = g$. Let $f(A) = H$. Diagram:



We showed above that for any subgroup H of a group G, there are homomorphisms $s, t : G \to Q$ such that $H = \{x \in G | s(x) = t(x)\}$. With this s and t, we have: $s|g(B) = t|g(B) \Rightarrow g(B) \subseteq f(A)$. This fact plus the injectivity of f imply that $h = f^{-1}g$ is well-defined. It's easy to check that this h is a homomorphism and the unique solution to $fh = g$. So f is an equalizer.

2 Adjoints

2.1 Units and Counits

Leinster simply asserts that units and counits are natural transformations without explaining why. Two diagrams help.

I omit parentheses when I can get away with it (e.g., FA' instead of $F(A')$). Assume $(F : \mathcal{A} \to \mathcal{B}) \dashv (G : \mathcal{B} \to \mathcal{A})$, $p : A' \to A$. Note that $\overline{1_{FA'}} = \eta_{A'}$.

The naturality requirement for adjunctions says that:

$$\begin{array}{ccc} 1_{FA'} \in \mathcal{B}(FA', FA') & \xrightarrow{\;\overline{}\;} & \mathcal{A}(A', GFA') \ni \eta_{A'} \\ {\scriptstyle Fp\circ\_}\big\downarrow & & \big\downarrow{\scriptstyle GFp\circ\_} \\ Fp \in \mathcal{B}(FA', FA) & \xrightarrow{\;\overline{}\;} & \mathcal{A}(A', GFA) \end{array}$$

Conclusion: $\overline{Fp} = GFp \circ \eta_{A'}$. Next use $\overline{1_{FA}} = \eta_A$.

$$\begin{array}{ccc} 1_{FA} \in \mathcal{B}(FA, FA) & \xrightarrow{\;\overline{}\;} & \mathcal{A}(A, GFA) \ni \eta_A \\ {\scriptstyle \_\circ Fp}\big\downarrow & & \big\downarrow{\scriptstyle \_\circ p} \\ Fp \in \mathcal{B}(FA', FA) & \xrightarrow{\;\overline{}\;} & \mathcal{A}(A', GFA) \end{array}$$

Conclusion: $\overline{Fp} = \eta_A \circ p$. Grand conclusion: $GFp \circ \eta_{A'} = \eta_A \circ p$, so $\eta : 1 \Rightarrow GF$ is a natural transformation. Duality gives the same for ε.

2.2 Free Monoids and Groups

(See also Exercise 2.2.12(b), §2.11.)

Leinster introduces the free functor from **Set** to **Group** in Example 1.2.4 (p.19) and again in Example 2.1.3(b) (p.44), calling it "tricky to construct explicitly". The free functor from **Monoid** to **Group** appears in Example 2.1.3(d) (p.45), where he says it is "again, tricky to describe explicitly". I'm not sure if he ever mentions the free functor from **Set** to **Monoid** in so many words. In this section (and *only* in this section), I will decorate F and U with subscripts as follows: $F_{\rm g}$ and $U_{\rm g}$ for the functors between **Set** and **Group**; $F_{\rm m}$ and $U_{\rm m}$ between **Set** and **Monoid**; unadorned F and U between **Monoid** and **Group**. (Elsewhere I think context makes it clear which functors are intended.)

Categories	**Unit** η_A	**Counit** ε_B
$\begin{CD} FA @>>> B \\ @A{F}AA @VV{G}V \\ A @>>> GB \end{CD}$	$\begin{CD} FA @>>> FA \\ @A{F}AA @VV{G}V \\ A @>>{\eta_A}> GFA \end{CD}$	$\begin{CD} FGB @>{\varepsilon_B}>> B \\ @A{F}AA @VV{G}V \\ GB @>>> GB \end{CD}$
$\mathbf{Vect} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{Set}$	UF(set) liberation (injective)	FU(vector space) evaluation (surjective)
$\mathbf{Set} \underset{D}{\overset{C}{\leftrightarrows}} \mathbf{LocConn}$	DC(loc conn top space) components (surjective)	CD(set) **iso**
$\mathbf{Top} \underset{U}{\overset{D}{\leftrightarrows}} \mathbf{Set}$	UD(set) **id**	DU(top space) discretization (bijective)
$\mathbf{Set} \underset{I}{\overset{U}{\leftrightarrows}} \mathbf{Top}$	IU(top space) indiscretization (bijective)	UI(set) **id**
$\mathbf{Abelian} \underset{U}{\overset{A}{\leftrightarrows}} \mathbf{Group}$	UA(group) abelianization (surjective)	AU(abelian group) **iso**

Figure 1: Units and Counits

Categories	**Unit** η_A	**Counit** ε_B
$\begin{array}{ccc} FA & \longrightarrow & B \\ F\uparrow & & \downarrow G \\ A & \longrightarrow & GB \end{array}$	$\begin{array}{ccc} FA & \longrightarrow & FA \\ F\uparrow & & \downarrow G \\ A & \xrightarrow[\eta_A]{} & GFA \end{array}$	$\begin{array}{ccc} FGB & \xrightarrow{\varepsilon_B} & B \\ F\uparrow & & \downarrow G \\ GB & \longrightarrow & GB \end{array}$
Group $F\uparrow\;\downarrow U$ **Monoid**	UF(monoid) liberation	FU(group) **iso**
Monoid $U\uparrow\;\downarrow R$ **Group**	RU(group) **id**	UR(monoid) invertible elements (injective)
G-set $F\uparrow\;\downarrow U$ **Set**	UF(set) liberation (injective)	FU(G-set) evaluation (surjective)
Set $U\uparrow\;\downarrow \mathrm{Map}$ **G-set**	Map(U(G-set)) (injective)	U(Map(set)) (surjective)
Set $\mathrm{Orb}\uparrow\;\downarrow \mathrm{Triv}$ **G-set**	Triv(Orb(G-set)) orbits (surjective)	Orb(Triv(set)) **iso**
G-set $\mathrm{Triv}\uparrow\;\downarrow \mathrm{Fix}$ **Set**	Fix(Triv(set)) **id**	Triv(Fix(G-set)) fixed points (injective)

Figure 2: More Units and Counits

The free group on a monoid has been called the *universal enveloping group* or the *group completion.*

$F_{\rm m}$ is a piece of cake: call the elements of a set X *letters*; then $F_{\rm m}X$ is just the set of all finite strings of letters, with concatenation for multiplication. (The empty string is the identity; I'll denote it by ϵ.) $F_{\rm g}$ has nearly the same description: we introduce two letters for each $x \in X$, denoted x and x^{-1}, and consider all finite strings of letters. A string is *reduced* if no pair xx^{-1} or $x^{-1}x$ occurs in it. It seems obvious that: (a) the set for $F_{\rm g}X$ is the set of all reduced strings; (b) any string of letters has a unique reduction; (c) multiplication in $F_{\rm g}X$ is concatenation followed by reduction, i.e., we concatenate the strings and then carry out all possible cancellations; (d) this multiplication is associative. But proving (a)–(d) formally looks like a bit of a drag. Bergman [2, §3.4] gives a slick proof of (b) and (d), due to van der Waerden. Let A be the set of reduced strings, and with any $x \in X$ associate a permutation of σ_x of A, as follows:

$$\sigma_x(x^{-1}t) = t \text{ for strings beginning with } x^{-1}$$
$$\sigma_x(t) = xt \text{ for strings not beginning with } x^{-1}$$

It's a pleasant exercise to verify the following: (e) For each $x \in X$, σ_x is a permutation on A. (f) If we associate $(\sigma_x)^{-1}$ with x^{-1}, then $\sigma_{x^{-1}}(\epsilon) = x^{-1}$. Now define a mapping $s \mapsto \sigma_s$ for any string of letters $s = l_1 \cdots l_n$ by setting $\sigma_s = \sigma_{l_1} \cdots \sigma_{l_n}$, with composition of permutations on the right. (g) If s is reduced, then $\sigma_s(\epsilon) = s$. (h) If s' is a reduction of s, then $\sigma_{s'} = \sigma_s$. (i) From (g) and (h) we can conclude (b). (j) Letting $s \cdot t$ denote the multiplication on A as defined in (c), we have $\sigma_{s \cdot t} = \sigma_s \sigma_t$. (k) The mapping $s \mapsto \sigma_s$ is a monomorphism of A into $\mathrm{Symm}(A)$, showing that A is isomorphic to a subgroup of a symmetric group, and thus verifying (d). (l) A is the free group on X, i.e., satisfies the required universal property.

Bergman provides another construction, where $F_{\rm g}X$ is a subgroup of a direct product. (Lang [5, §1.12] gives much the same construction.) Let S be a set of cardinality $\max(\aleph_0, |X|)$. Let $\{G_i | i \in I\}$ be an indexing of all groups whose underlying sets are subsets of S. Let $G = \prod_{i\in I}\prod_{f:X\to G_i} G_i$, so we have as many copies of G_i in the product as there are ways to map X into G_i. For convenience, we'll write $G = \prod_{i,f} G_{if}$, so all the G_{if}'s are equal to G_i. (Technically, each f maps X into $U_\mathrm{g}G_i$, but I'll omit all mention of U_g for most of this paragraph, to reduce clutter without, hopefully, sowing any confusion.) For any $x \in X$, we define $\eta(x) \in G$ by letting the (if) component of $\eta(x)$ be $f(x)$: $\eta(x)_{if} = f(x)$. Let H be the subgroup of G generated by the set $\eta(X) = \{\eta(x)|x \in X\}$. H is $F_\mathrm{g}X$. Proving that H has the required universal property is a breeze. Suppose $h : X \to K$ for some group K. Let L be the subgroup of K generated by $h(X)$. The cardinality of L is no more than $\max(\aleph_0, |X|)$, so we might as well assume that L is one of the G_i's. Then h must be one of the f's for that G_i (or technically $f : X \to G_{if} = L$ is a range restriction of $h : X \to K$). By the definition of $\eta(x)$, we have $h(x) = f(x) = \eta(x)_{if} = p_{if}(\eta(x))$, where $p_{if} : H \to G_{if}$ is the projection map $G \to G_{if}$ restricted to H. So p_{if} composed with the inclusion $\iota : G_{if} \hookrightarrow K$ completes the commutative diagram, and it's easy to see that $\iota \circ p_{if}$ is the unique such group homomorphism (because h specifies the values for all $x \in X$, and $\eta(X)$ and $h(X)$ generate H and $G_{if} = L$ respectively). (Bergman's treatment offers more motivation than I did.) Note that η is, sort of, the unit of the adjunction $F_\mathrm{g} \dashv U_\mathrm{g}$; "sort of" because I've been cavalier about whether η maps X into $F_\mathrm{g}X$ or into its underlying set $U_\mathrm{g}F_\mathrm{g}X$; the unit of course maps $X \to U_\mathrm{g}F_\mathrm{g}X$.

We take another look at this construction in §6.13, relating it to the General Adjoint Functor Theorem (GAFT).

Before we discuss F, it will help to define presentations of groups and monoids.

A presentation of a group consists of a set of generators plus relations; for example, $\langle x, y, z|xy = yx, zz = x^{-1}\rangle$. More precisely, say X is a set and $F_\mathrm{g}X$ is its free group; a relation is a pair (s, t) with $s, t \in F_\mathrm{g}X$, thought of as a condition $s = t$ to be satisfied in the presented group. Now say we have a presentation $\langle X|R\rangle$, where R is a set of relations. We recast

$s = t$ as $st^{-1} = 1$, and let N be the least normal subgroup generated by $\{st^{-1}|(s,t) \in R\}$, i.e., the intersection of all normal subgroups of $F_{\text{g}}X$ containing all the st^{-1}'s. Then $F_{\text{g}}X/N$ is the group presented by $\langle X|R\rangle$.

Alternately, the direct product construction of $F_{\text{g}}X$ is readily adapted to incorporate relations.

We need to proceed differently for a presentation of a monoid. Again we start with a set X of generators and form the free monoid $F_{\text{m}}X$, but we no longer have the option of rewriting relations $s = t$ as $st^{-1} = 1$. Instead, we invoke the notion of a *congruence*: that's an equivalence relation $\equiv$ on a monoid M with the property that if $s \equiv t$ for $s, t \in M$, then $rs \equiv rt$ and $sr \equiv tr$ for all $r \in M$. (It's enough to check this with r ranging over a set of generators for M.) If $\equiv$ is a congruence on M, then monoid multiplication is well-defined on the set of congruence classes of M, turning $M/\equiv$ into a monoid—the *quotient* monoid, of course! The natural map $M \to M/\equiv$ is a monoid morphism. OK, given a presentation $\langle X|R\rangle$, with $s, t \in F_{\text{m}}X$ for all $(s,t) \in R$, we can form the least congruence $\equiv$ such that $s \equiv t$ for all the relations in R. Since equivalence relations are sets of ordered pairs, this least congruence is the intersection of all congruences containing R. But it can also be described by a more explicit "bottom up" construction; a slight modification of Remark 5.2.8 (p.129) does the trick. $F_{\text{m}}X/\equiv$ is the monoid presented by $\langle X|R\rangle$.

Now we are ready to define FM, the free group generated by a monoid M. This is the group with presentation

$$\langle M|\{(a, bc)|a = bc \text{ in } M, a, b, c \in M\}\rangle$$

in other words, the generators are the elements of M, and the set of relations is the complete multiplication table of M. Showing that this enjoys the defining universal property is not difficult. Here's the diagram, as a reminder:

$$\begin{array}{ccccc} M & \xrightarrow{\eta_M} & UFM & \qquad & FM \\ & \searrow^{\forall f} & \downarrow{\scriptstyle U\bar{f}} & & \downarrow{\scriptstyle \exists!\bar{f}} \\ & & UG & & G \end{array}$$

While congruences aren't needed to define FM, they're handy to have around. For one thing, if $\langle X, R\rangle$ is a monoid presentation for M, then it's *also* a group presentation for FM.

Now let's look at the unit $\eta = UF$ and counit $\varepsilon = FU$. First observe that ε_G is the identity (or at least an isomorphism) for all G: if a monoid already possesses all inverses, so it "is" a group (albeit domiciled in **Monoid**), then regarded as a group, it satisfies the universal property for FM. (This is easily checked.)

In the reverse direction, is η_M injective? Experience with F_{g} and F_{m} might suggest it is. Alas, no. First, if M does not satisfy the cancellation laws:

$$(\forall a, b, c \in M)[ab = ac \Rightarrow b = c]$$
$$(\forall a, b, c \in M)[ba = ca \Rightarrow b = c]$$

then η_M will not be injective, since these laws follow immediately from the existence of inverses. As a concrete example, let X be a set with two elements, and let M be the monoid of all functions $X \to X$. Easy exercise: FM is the trivial group, η_M is the morphism to the trivial monoid, η_M is not monic, and F sends all five morphisms in $\mathbf{Monoid}(M, M)$ to the morphism 1_{FM}, demonstrating that F is neither full nor faithful. (Hint: M has four elements: the identity 1, two constant functions, and a function that swaps the two elements. If a is one of the constant functions, then $am = a1$ for all $m \in M$. Now use cancellation in FM.)

If M satisfies the cancellation laws, is η_M then injective? Not always. Malcev came up with an infinite family of laws that all together are necessary and sufficient for η_M to be injective (see Cohn [3, §VII.3]). Moreover, no finite subfamily of the Malcev laws suffice. Here is the next simplest law in this family:

$$\begin{gathered}(\forall a, b, c, d, e, f, g, h \in M) \\ [ab = cd \;\&\; ce = af \;\&\; gf = he \Rightarrow gb = hd]\end{gathered} \tag{*}$$

According to Cohn, the monoid with the presentation

$$\langle a, b, c, d, e, f | ab = cd, ce = af, gf = he\rangle$$

satisfies the cancellation laws but not (*). It's easy to show that any group satisfies (*), though:

$$\begin{aligned} gb &= ga^{-1}ab = ga^{-1}cd = ga^{-1}cee^{-1}d \\ &= ga^{-1}afe^{-1}d = gfe^{-1}d = hee^{-1}d = hd \end{aligned}$$

So for the monoid with the presentation above, η_M sends the distinct elements gb and hd to the same element of $\eta_M M$.

I think this justifies Leinster's use of "tricky".

2.3 2.1.13, p.49

The discrete categories have a bijection between their sets of objects. Proof: $|\mathcal{A}(FA, B)| = |\mathcal{B}(A, GB)|$, so $FA = B$ iff $A = GB$ because the categories are discrete. This means that $F = G^{-1}$.

2.4 2.1.15, p.50

Since I is initial, $|\mathcal{A}(I, A)| = 1$ for all $A \in \mathcal{A}$. In particular, $|\mathcal{B}(FI, B)| = |\mathcal{A}(I, GB)| = 1$ for all $B \in \mathcal{B}$, so FI is initial.

2.5 2.1.16(a), p.50

Let A be a G-set. First we list functors from $[G, \mathbf{Set}]$ to $\mathbf{Set}$. (1) UA is the underlying set A, i.e., forget the action. (2) $\mathrm{Orb}(A)$ is the set of orbits of A under the action. (3) $\mathrm{Fix}(A)$ is the set of fixed points under the action. Defining the effect on morphisms (i.e., G-equivariant maps), and checking functoriality, is routine for all of these.

Next, functors from $\mathbf{Set}$ to $[G, \mathbf{Set}]$. Let X be any set. (4) Define the action trivially, leaving every element of X fixed; call this $\mathrm{Triv}(X)$ (5) Define an action on $G \times X$ by $g \cdot (h, x) = (gh, x)$. Call this FX, so $UF(X) = G \times X$. F is the "free" functor: we think of (g, x) as representing $g \cdot x$ as a "formal expression". (6) Let X^G be the set of functions from G to X, and define the action by $(g \cdot u)(k) = u(kg)$ for all $u : G \to X$ and all $k \in G$. Call this $\mathrm{Map}(X)$. We check that this is a left-action[2]:

$$(g \cdot (h \cdot u))(k) = (h \cdot u)(kg) = u(kgh) = (gh \cdot u)(k)$$

Defining the effect on morphisms and checking functoriality is trivial for Triv. With F, we set $(Ff)(g, x) = (g, f(x))$ for $f : X \to Y$; the required checks are easy. With Map, we define $(\mathrm{Map}\, f)(u)$ as the composition $G \xrightarrow{u} X \xrightarrow{f} Y$. Again the checks are easy.[3]

$F \dashv U$. If $r : X \to UB$ where B has a G-action defined on it, define $\overline{r} : FX \to B$ by $\overline{r}(g, x) = g \cdot r(x)$. Easy to check that this is G-equivariant. For $s : FX \to B$, define $\overline{s} : X \to UB$ by $\overline{s}(x) = s(1, x) \in UB$: technically $s(1, x) \in B$, but we can regard it as an element of UB without harm. To borrow some jargon from computer science, we "cast" elements of B to elements of UB, without loss of information. Easy to check that $\overline{\overline{r}} = r$ and $\overline{\overline{s}} = s$. Finally, naturality: if $p : X' \to X$ and $q : B \to B'$, we need to check

[2]This verification would not have worked if we had used the seemingly more natural definition $(g \cdot u)(k) = u(gk)$.

[3]Defining Map with G^X instead of X^G results in a contravariant functor; this problem concerns only covariant functors.

that $\overline{rp} = \overline{r}F(p)$ and $\overline{qs} = U(q)\overline{s}$. Well,

$$\begin{aligned}\overline{rp}(g, x') &= g \cdot rp(x') = g \cdot r(px') \\ &= \overline{r}(g, px') = \overline{r}(F(p)(g, x')) \\ &= (\overline{r}F(p))(g, x')\end{aligned}$$

and

$$\overline{qs}(x) = qs(1, x) = U(q)\overline{s}(x)$$

where the $U(q)$ appears because we've cast $s(1, x)$ from B to UB.

$U \dashv \mathrm{Map}$. Let $r : A \to \mathrm{Map}(Y)$. This time, A has the action and Y is a plain set. Define $\overline{r} : UA \to Y$ by $\overline{r}(a) = (ra)(1)$, where 1 is the identity of G. (If we picture $ra : G \to Y$ as a G-indexed tuple, then $\overline{r}(a)$ is the component at 1.)

Let $s : UA \to Y$. Define $\overline{s} : A \to \mathrm{Map}(Y)$ by first setting $s_a(k) = s(k \cdot a)$ for all $k \in G$. So $s_a \in Y^G$. We then set $\overline{s}(a) = s_a$, making $\overline{s}$ a mapping from A to $\mathrm{Map}(Y)$.

We describe s_a another way. The mapping $o_a : k \mapsto k \cdot a$ (with $k \in G$) is related to the orbit of a in A—in fact, the orbit is the range of o_a. Picture o_a as a G-indexed tuple of A-elements. Then s_a is just the composition $G \xrightarrow{o_a} UA \xrightarrow{s} Y$. In other words, we apply s to the tuple component-by-component. (Also we've cast the codomain of o_a from A to UA.)

We verify that $\overline{s}$ is equivariant. The requirement of equivariance is summed up in the "rectangle" diagram:

$$\begin{array}{ccccc} a & \mapsto & o_a & \mapsto & s_a \\ g \cdot a & \mapsto & o_{g \cdot a} & \mapsto & s_{g \cdot a} \stackrel{?}{=} g \cdot s_a \end{array}$$

and the bottom right equation checks out:

$$\begin{array}{ccccccccc} o_{g \cdot a}(k) & = & k \cdot (g \cdot a) & = & kg \cdot a & = & o_a(kg) & = & (g \cdot o_a)(k) \\ s_{g \cdot a}(k) & = & s(k \cdot (g \cdot a)) & = & s(kg \cdot a) & = & s_a(kg) & = & (g \cdot s_a)(k) \end{array}$$

Note that $\overline{s}(a)(1) = s_a(1) = s(1 \cdot a) = s(a)$. That is, the s_a tuple has component $s(a)$ at index 1. (Because the o_a tuple has component a at index 1.) But $\overline{s}(a)(1)$ is just the definition of $\overline{\overline{s}}(a)$. So $\overline{\overline{s}} = s$.

As for $\overline{\overline{r}}$, write $r_a = r(a)$, so r_a is a G-indexed tuple, and $\overline{r}(a) = r_a(1)$. Since r is equivariant, we have for any $g, k \in G$

$$\begin{aligned} r_{g\cdot a} &= g \cdot r_a \\ r_{g\cdot a}(k) &= (g \cdot r_a)(k) = r_a(kg) \\ r_{g\cdot a}(1) &= r_a(g) \quad \text{letting } k = 1 \end{aligned}$$

Now we compute $\overline{\overline{r}}(a)(g)$ for arbitrary $a \in A$, $g \in G$:

$$\begin{aligned} \overline{\overline{r}}(a)(g) &= \overline{r}(g \cdot a) \\ &= r_{g\cdot a}(1) = r_a(g) = r(a)(g) \end{aligned}$$

so $\overline{\overline{r}} = r$.

Naturality: let $p : A' \to A$ and $q : Y \to Y'$. We need to show that $\overline{rp} = \overline{r}Up$ and $\overline{qs} = (\mathrm{Map}\, q)\overline{s}$. The proof just involves unwinding the definitions. A good deal of currying goes on (e.g., the definition of $\overline{s}$ as $\overline{s} : a \mapsto s_a$ with $s_a : k \mapsto s(k \cdot a)$), but the verifications consist of filling in all the arguments and turning the crank. So to show $\overline{rp} = \overline{r}Up$, we show $\overline{rp}(a') = \overline{r}Up(a')$ for all $a' \in A'$—both sides work out to $r_{pa'}(1)$, with our convention that $r(a) = r_a$. To show $\overline{qs} = (\mathrm{Map}\, q)\overline{s}$, we show $\overline{qs}(a)(k) = ((\mathrm{Map}\, q)\overline{s})(a)(k)$ for all $a \in A$, $k \in G$. Here both sides work out to $qs(k \cdot a)$.

Orb $\dashv$ Triv. Let $r : A \to \mathrm{Triv}(Y)$ and $s : \mathrm{Orb}(A) \to Y$. Because r is equivariant and G acts trivially on $\mathrm{Triv}(Y)$, r must send entire orbits in A to single elements in $\mathrm{Triv}(Y)$. Of course s does the same. The only distinction between r and s is that r "sees" the individual elements of an orbit, but treats them all the same; s "sees" an orbit as a single thing. The definitions of $\overline{r}$, $\overline{s}$, their duality, and the naturality equations are all very direct.

Triv $\dashv$ Fix. Let $r : X \to \text{Fix}(B)$ and $s : \text{Triv}(X) \to B$, with B having the action and X being plain. Because s is equivariant and G acts trivially on $\text{Triv}(X)$, the image of s must consist entirely of fixed elements. Abusing notation, we could write $s(X) \subseteq \text{Fix}(B)$. Not abusing notation, we write $\overline{s} : X \to \text{Fix}(B)$. In the reverse direction, since r maps X to the fixed elements of B, r is equivariant when regarded as mapping $\text{Triv}(X)$ to B; of course, it's really $\overline{r}$ that maps $\text{Triv}(X)$ to B. (The distinction between r and $\overline{r}$ being a range restriction plus casting.) All the verifications are easy.

See also §6.2 and §6.9(c).

2.6 2.1.16(b), p.50

Let V be a vector space, and let $\widetilde{V}$ be V with a G-action on it. Fix, Triv, U, F, and Map all have analogs. $\text{Fix}(\widetilde{V})$ is the subspace left fixed by the action; $\text{Triv}(V)$ is V with a trivial action; $U(\widetilde{V})$ is the underlying vector space; $F(V)$ consists of all formal finite sums $\sum g_i v_i$ subject to the conditions $g(v + w) = gv + gw$. In other words, we find the subspace generated by the formal sums $g(v+w)+g(-v)+g(-w)$, and divide out by it. (Variations in this definition are possible, leading to isomorphic representations.) Finally, we make V^G into a vector space by defining, for $u, v \in V^G$ and a a scalar, $(u + v)(k) = u(k) + v(k)$ and $(au)(k) = a(u(k))$ (for all $k \in G$). The G-action (defined as before) is linear. (Thinking of V^G as a vector space of G-indexed tuples makes all the verifications pretty obvious.)

It is routine to check that Triv $\dashv$ Fix and $F \dashv U \dashv$ Map.

However, we lose Orb. The orbits of $\widetilde{V}$ do not form a subspace. Simple example: let $V = \mathbb{R}^2$ under the action of the rotation group $SO(2)$. The orbits are circles, plus the origin. The sum $gv + hw$ can lie on any circle between the circles for $v + w$ and $v - w$ (assuming $|v| \geq |w|$).

2.7 2.1.17, p.50

First I describe the functors, then worry about the adjunctions. I write **Presheaf**(X) for the category of presheaves on the topological space X, i.e., the functor category $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$. For this problem, I'll let F and G be objects of **Presheaf**(X) and I'll let A and B be sets, with $f : A \to B$ a function. When $Y \supseteq Z$ are open subsets of X, there's a unique morphism in $\mathcal{O}(X)^{\text{op}}$; call it $Y \overset{\supseteq}{\Rightarrow} Z$. Write just F_{YZ} for $F(Y \overset{\supseteq}{\Rightarrow} Z)$.

We need to know what morphisms in **Presheaf**(X) look like. As a functor category, the morphisms are natural transformations. So a morphism $\alpha : F \Rightarrow G$ is a family of maps $\alpha_Y : FY \to GY$ making these diagrams commute for all $Y \overset{\supseteq}{\Rightarrow} Z$:

$$\begin{array}{ccc} FY & \xrightarrow{\alpha_Y} & GY \\ {\scriptstyle F_{YZ}}\downarrow & & \downarrow{\scriptstyle G_{YZ}} \\ FZ & \xrightarrow[\alpha_Z]{} & GZ \end{array}$$

I'll use α for such a morphism.

$\Lambda : \mathbf{Set} \to$ **Presheaf**(X): We need to describe a presheaf ΛA for each set A, and a morphism of presheaves Λf for each function $f : A \to B$. Let $\Lambda A(\varnothing) = A$ and let $\Lambda A(Y) = \varnothing$ for each $Y \neq \varnothing$. The morphisms in ΛA are all empty functions except for 1_A on $\Lambda A(\varnothing)$. Letting $\Lambda f = \alpha$, we need to define $\alpha_Y : \Lambda A(Y) \to \Lambda B(Y)$ for each Y. Let $\alpha_\varnothing = f$ and let α_Y be the empty function for all $Y \neq \varnothing$.

$\Pi :$ **Presheaf**$(X) \to \mathbf{Set}$: We need to define a set ΠF for any F, and a function $\Pi\alpha$ for any morphism α. Let $\Pi F = F\varnothing$. Let $\Pi\alpha = \alpha_\varnothing$. The functoriality equation $\Pi(\alpha\beta) = \Pi(\alpha)\Pi(\beta)$ follows from the definition of composition of natural transformations, which says that $(\alpha\beta)_\varnothing = \alpha_\varnothing\beta_\varnothing$.

$\Delta : \mathbf{Set} \to$ **Presheaf**(X): As defined in the problem statement, ΔA is the constant functor $\Delta A(Y) = A$ for all Y and $\Delta A_{YZ} = 1_A$ for all $Y \overset{\supseteq}{\Rightarrow} Z$.

Γ : **Presheaf**$(X) \to$ **Set**: Similar to Π. Let $\Gamma F = FX$. Let $\Gamma\alpha = \alpha_X$.

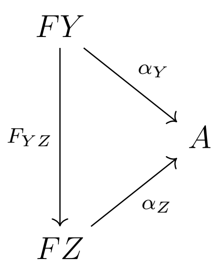
∇ : **Set** $\to$ **Presheaf**(X): Similar to Λ. Let $\nabla A(X) = A$ and let $\nabla A(Y) = 1$ for each $Y \neq X$, where 1 is some arbitrary singleton. The morphisms in ∇A are all "constantly 1" functions except for 1_A on $\nabla A(X)$. Setting $\nabla f = \alpha$ for $f : A \to B$, we let $\alpha_X = f$ and let α_Y be 1_1 for all $Y \neq X$.

One more functor from **Set** to **Presheaf**(X): $\Phi A(Y)$ is the set of A-valued functions with domain Y, and the morphisms are all restriction maps. (If we give A a topology, we can ask that all the functions be continuous. With the indiscrete topology, this is automatic.) I'm disappointed that this functor doesn't show up in this problem.

We have to show that $\Lambda \dashv \Pi \dashv \Delta \dashv \Gamma \dashv \nabla$.

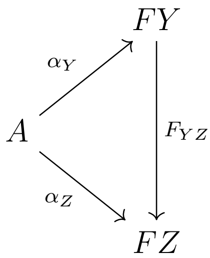
$\Lambda \dashv \Pi$. Suppose $\alpha : \Lambda A \Rightarrow F$. Since $\Lambda A(Y) = \varnothing$ for all $Y \neq \varnothing$, all such $\alpha_Y : \Lambda A(Y) \to FY$ are empty functions. So α is completely determined by $\alpha_\varnothing$, which is a function $A \to F\varnothing$, and $\alpha_\varnothing$ can be any such function. But $\Pi F = F\varnothing$, so $\alpha_\varnothing$ is an arbitrary function $A \to \Pi F$. It is now clear that $\Lambda \dashv \Pi$ except for a routine check of the naturality equations.

$\Pi \dashv \Delta$. Suppose $\alpha : F \Rightarrow \Delta A$. So for every Y, $\alpha_Y : FY \to \Delta A(Y) = A$. The naturality squares become triangles:



So α_Y is determined by α_Z, and in particular, all the α's are determined by $\alpha_\varnothing$. Now we compare with $f : \Pi F \to A$, i.e., $f : F\varnothing \to A$, and we have the desired 1–1 correspondence. All verifications are routine.

$\Delta \dashv \Gamma$. Suppose $\alpha : \Delta A \Rightarrow F$. Similar to $\Pi \dashv \Delta$—the naturality squares become triangles:



So α is completely determined by α_X, and α_X can be any function $A \to FX = \Gamma F$. The rest is straightforward.

$\Gamma \dashv \nabla$. Very similar to $\Lambda \dashv \Pi$. Let $\alpha : F \Rightarrow \nabla A$. All α_Y except for α_X are uniquely determined as maps to a singleton, and α_X can be any function $FX \to \nabla A(X) = A$. Routine verifications.

See also §6.10.

2.8 2.2.10, p.56

Suppose $f(a) \leq b \leftrightarrow a \leq g(b)$. Now, $f(a) \leq f(a)$, so $a \leq g(f(a))$ (applying the $\to$ implication). And $g(b) \leq g(b)$, so $f(g(b)) \leq b$ (applying the $\leftarrow$ implication).

Suppose $a \leq g(f(a))$ and $f(g(b)) \leq b$, with f and g order preserving. Suppose $f(a) \leq b$. Apply g to both sides: $g(f(a)) \leq g(b)$, so $a \leq g(f(a)) \leq g(b)$. Suppose $a \leq g(b)$. Apply f to both sides, so $f(a) \leq f(g(b)) \leq b$.

2.9 2.2.11, p.57

For (a), we have to show that F sends $\mathcal{A}' = \mathrm{Fix}(GF)$ into $\mathcal{B}' = \mathrm{Fix}(FG)$ and vice versa for G. The definition of equivalence (1.3.15, p.34) is then satisfied. For $A \in \mathcal{A}'$, we have $\eta_A : A \cong GFA$, so $F\eta_A : FA \cong FGFA$. That is, FA is isomorphic to $FG(FA)$ via $F\eta_A$. The triangle identity says that $F\eta_A\varepsilon_{FA} = 1_{FA}$, so $\varepsilon_{FA} = (F\eta_A)^{-1}$. Therefore ε_{FA} is an isomorphism.

For (b), Figs.1 and 2 give us plenty to work with. Start with the cases where a unit or counit is marked 'id' or 'iso'. (The latter means "canonically iso", e.g., $\mathrm{Orb}(\mathrm{Triv}(x)) = \{x\}$.) This means that either $\mathcal{A}' = \mathcal{A}$ (unit=id/iso) or $\mathcal{B}' = \mathcal{B}$ (counit=id/iso). $D \dashv U$ for **Set** and **Top** is typical: $\mathbf{Set}' = \mathbf{Set}$, but $\mathbf{Top}'$ are the discrete spaces. ($DU(X) \to X$ is continuous for any space X, but is an isomorphism only for discrete spaces.)

Let's go down the list, looking for isos. For the pair C, D, $\mathbf{LocConn}'$ are discrete spaces. For U, I, $\mathbf{Top}'$ are indiscrete spaces. For A, U, $\mathbf{Group}'$ are the abelian groups. For F, U on monoids and groups, $\mathbf{Monoid}'$ are groups. Likewise with the pair U, R. For both (Orb, Triv) and (Triv, Fix) we have $\mathbf{G-set}'$ being the G-sets where G acts trivially.

Now for the non-iso cases. The $F \dashv U$ pair for **Vect** and **Set** is kind of strange. $FU(V)$ has dimension bigger than V, even when V is the zero space. $\eta_S : S \to UF(S)$ is not surjective for any S. So it looks like $\mathbf{Vect}'$ and $\mathbf{Set}'$ are both empty categories.

It's the same story for $F \dashv U$ with **G-set** and **Set**, except when $G = 1$. In that case, $\mathbf{G-set}' = \mathbf{G-set}$ and $\mathbf{Set}' = \mathbf{Set}$, and the equivalence comes from forgetting and remembering the trivial action of G, which is the only possible one.

2.10 2.2.12(a), p.57

Assume $\mathcal{A}\ \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}}\ \mathcal{B}$. Using the notions of §1.11, we can state a stronger result.

1. G is faithful iff ε_B is epic for all $B \in \mathcal{B}$. (Dually, F is faithful iff η_A is monic for all $A \in \mathcal{A}$.)
2. G is full iff ε_B is split monic for all $B \in \mathcal{B}$. (Dually, F is full iff η_A is split epic for all $A \in \mathcal{A}$.)

As noted in §1.11, if g is both split monic and epic, then g is an isomorphism. With this fact, we see that (1) and (2) imply the exercise. (I leave the dual versions as an exercise. Or just say "duality!" in a dramatic tone of voice.)

It helps to bear the naturality diagram in mind:

$$\begin{array}{ccccc} FGB & \xrightarrow{\varepsilon_B} & B & \xrightarrow{q} & B' \\ {\scriptstyle F}\uparrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G} \\ GB & \xrightarrow{1_{GB}} & GB & \xrightarrow{Gq} & GB' \end{array}$$

This means we have a 1–1 correspondence between the $q\varepsilon_B$ morphisms and the $Gq1_{GB} = Gq$ morphisms. For G to be faithful, we need $q \mapsto Gq$ to be 1–1; for ε_B to be epic, we need $q \mapsto q\varepsilon_B$ to be 1–1. This is the essence of the faithful/epic equivalence. For an algebraic proof (next two paragraphs), we use the equations $\overline{\varepsilon_B} = 1_{GB}$ and $\overline{q\varepsilon_B} = Gq$ implied by the diagram.

Epic implies faithful: $Gq_1 = Gq_2 \Rightarrow \overline{q_1\varepsilon_B} = \overline{q_2\varepsilon_B} \Rightarrow q_1\varepsilon_B = q_2\varepsilon_B \Rightarrow q_1 = q_2$.

Faithful implies epic: $q_1\varepsilon_B = q_2\varepsilon_B \Rightarrow \overline{q_1\varepsilon_B} = \overline{q_2\varepsilon_B} \Rightarrow Gq_1 = Gq_2 \Rightarrow q_1 = q_2$.

Next, split monic implies full: the key again is the naturality diagram. Suppose $h : GB \to GB'$. We have to find a q with $Gq = h$. We can treat h as going from the leftmost GB to GB' on the right, along the bottom; then $\overline{h}$ will go from FGB to B', along the top. But we want q just to go from B to B'. A left inverse λ to ε_B can cancel out the ε_B portion of the journey, giving the desired $q : B \to B'$.

Here's the formal argument. Since ε_B is split monic, let $\lambda\varepsilon_B = 1_{FGB}$. Given $h : GB \to GB'$, set $\overline{h}\lambda = q : B \to B'$. Then $\overline{h}\lambda\varepsilon_B = \overline{h} = q\varepsilon_B$. Hence $h = \overline{q\varepsilon_B} = Gq$, as desired.

Full implies split monic: we want to find a λ making the composition $FGB \xrightarrow{\varepsilon_B} B \xrightarrow{\lambda} FGB$ the identity. Again we appeal to the naturality diagram, letting $q = \lambda$ and $B' = FGB$:

$$\begin{array}{ccccc} FGB & \xrightarrow{\varepsilon_B} & B & \xrightarrow{\lambda} & FGB \\ {\scriptstyle F}\uparrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G} \\ GB & \xrightarrow{1_{GB}} & GB & \xrightarrow{G\lambda} & GFGB \end{array}$$

We want $\overline{\lambda\varepsilon_B} = \overline{1_{FGB}} = \eta_{GB}$, and the diagram yields $\overline{\lambda\varepsilon_B} = G\lambda$, so using the fullness of G we choose a λ giving $G\lambda = \eta_{GB}$. We then have

$$\begin{aligned} \overline{\lambda\varepsilon_B} &= G\lambda = \eta_{GB} \\ \overline{1_{FGB}} &= \eta_{GB}, \text{ therefore} \\ 1_{FGB} &= \lambda\varepsilon_B \end{aligned}$$

as desired.

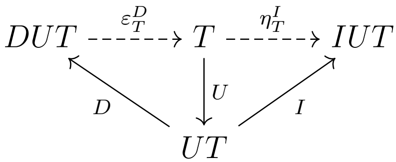
2.11 2.2.12(b), p.57

We go further than asked, and see how 2.2.12(a) applies, or doesn't, for most of figs.1 and 2. See §1.11 for background on monics and epics.

The faithfulness assumption isn't needed for the "split" implications, but that won't matter for this exercise. The assumption holds for all the categories in figs.1 and 2, since the morphisms are all "really" functions subject to conditions, namely homomorphisms, continuous maps, or equivariant maps.

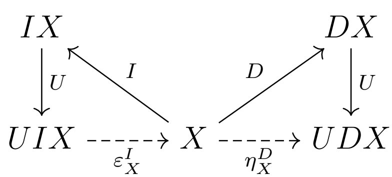
We begin with a chain of adjunctions.

Topological spaces: $D \dashv U \dashv I$. This diagram (*not* a commutative diagram) lays out the situation for the counit ε^D of $D \dashv U$ and the unit η^I of $U \dashv I$. Note that dashed arrows represent morphisms and solid arrows functors.



On the top, we have the topological space T sandwiched between the discrete space and the indiscrete space obtained from T. The morphisms are bijective—identities in **Set**—hence epic and monic respectively. This jibes with part (a) of this exercise, since U is faithful. Left inverses do not (usually) exist, since the set-theoretic inverses are not (usually) continuous. As expected, since U is not full—not all functions are continuous.

The diagram for the unit η^D and counit ε^I looks like this:

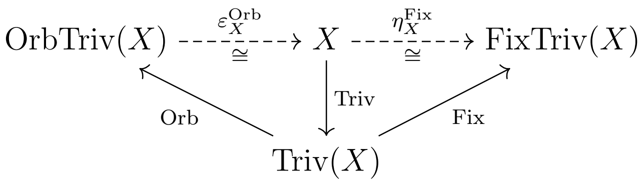


The unit and counit are both the identity function 1_X. Both D and I are full and faithful.

Locally connected topological spaces: $C \dashv D$. D is full and faithful, and the counit ε_X is an isomorphism. (Technically not the identity but the function $\{x\} \mapsto x$.) The unit η_T "collapses components", i.e., sends x to its containing component $[x]$. So η_X is (usually) not monic, and C isn't faithful (the "fine details" of f are lost in going to Cf).

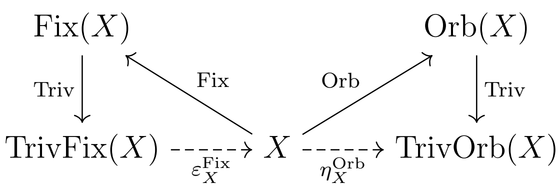
But η_T is split epic: let c_T be a choice function, with $c_T([x])$ choosing a point $x \in [x]$ for each component. Then $\eta_T \, c_T = 1_{DCT}$. C is full because given $g : CT \to CT'$ ($g : [x] \to g[x]$), we just set $f(x) = c_{T'}(g[x])$ for all $x \in T$ (so f sends entire components of T to single points in T').

G-sets: Orb $\dashv$ Triv $\dashv$ Fix. The counits of Orb and the units of Fix are isomorphisms:



Triv is full and faithful, so all's right with the world.

The units of Orb and the counits of Fix come out of this diagram:

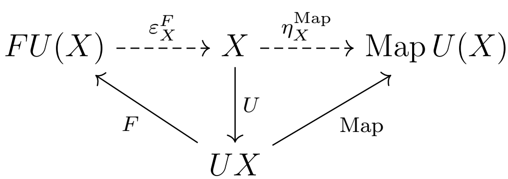


Orb is somewhat but not entirely similar to Components: η_X^{Orb} sends each x to its orbit $[x]$. So clearly injectivity fails, and it's no sweat to bootstrap this to a failure of monicity. Let G be transitive on X, so η_X^{Orb} sends X to a singleton. Now we just need a pair of distinct equivariant maps $f, g : X' \to X$, and we're home free. To show that Orb is not faithful, take G is transitive on X with two distinct equivariant maps $X \to X$.

Unlike Components, η_X^{Orb} is not always split epic and Orb is not full. Example: let X be a nonempty set, let X_1 be X with G acting trivially, and let X_2 be X with a G-action that acts without fixed points. (I.e., $(\forall x \in X)(\exists g \in G) g \cdot x \neq x$.) Then there are *no* equivariant maps from X_1 to X_2, but there is obviously at least one function from $\mathrm{Orb}(X_1)$ to $\mathrm{Orb}(X_2)$. So Orb isn't full. As for η_X^{Orb}, it's surjective and hence epic, but not split epic; $X = X_2$ provides an example.

As for Fix, it's neither faithful nor full. If $f : X \to X'$ is an equivariant map, then $\mathrm{Fix}(f)$ is the restriction of f to $\mathrm{Fix}(X)$. So $\mathrm{Fix}(f)$ tells you nothing about what's going on with the non-fixed points of X, demonstrating the lack of faith. Suppose G acts without fixed points on $X \neq \varnothing$. Then $\varepsilon_X^{\mathrm{Fix}}$ is the empty map $\varnothing \to X$, monic but not split monic. Also, there are no maps $X \to \varnothing$, but we have the empty function $\mathrm{Fix}(X) = \varnothing \to \varnothing$, showing the lack of fullness. (Mini-exercise: cook up an example with no equivariant maps $X \to X'$, but $\mathrm{Fix}(X) = \mathrm{Fix}(X') = \varnothing$.) Although not full, Fix is full on the full subcategory of **G-set** consisting of those X with $\mathrm{Fix}(X) \neq \varnothing$.

G-sets: $F \dashv U \dashv \mathrm{Map}$. For the unit of Map and the counit of F we look at this diagram:

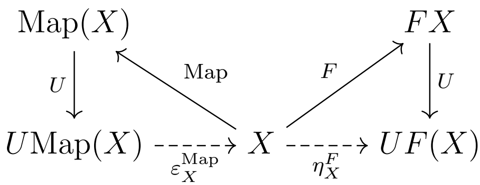


The unit η_X^{Map} is $x \mapsto o_x$, where o_x was defined in §2.5 as the G-indexed tuple $g \mapsto g \cdot x$. This is injective and hence monic, but generally not surjective and hence not split epic. (For an easy example, let X be finite and G nontrivial, so $|X^G| > |X|$.)

The theorem says that U must be faithful but not full. Easily checked directly.

The counit ε^F_X is $(g,x) \mapsto g \cdot x$. This is surjective and hence epic, but not injective and hence not split monic (unless G is trivial). The theorem says that U must be faithful but not full, duplicating what we already know from η^{Map}_X.

The unit of F and the counit of Map come out of this diagram:



The unit η^F_X is $x \mapsto (1,x)$. This is injective and hence monic, but not surjective and hence not split epic (unless G is trivial). So the theorem says that F must be faithful but not full. Faithfulness is obvious. To show that F is not full, we need an equivariant map $FX \to FY$ that is not Ff for any $f : X \to Y$. Now, $(Ff)(g,x) = (g, f(x))$. On the other hand, if $h \in G$ is not 1 and $p : X \to Y$ is arbitrary, then $(g,x) \mapsto (gh, p(x))$ is equivariant and not equal to Ff for any f.

The counit $\varepsilon^{\text{Map}}_X$ is $u \mapsto u(1)$, where $u \in X^G$ and 1 is the identity of G. This is surjective (because $o_x(1) = x$) and hence epic, but generally not injective and hence not split monic. (Again X finite and G nontrivial provides an easy example, via cardinalities.)

The theorem tells us that Map is faithful but not full. Faithfulness is easy: if $f_1, f_2 : X \to Y$ are distinct, say $f_1(x) \neq f_2(x)$, then if $u(1) = x$, we have $f_1 u \neq f_2 u$, i.e., $\text{Map} f_1 \neq \text{Map} f_2$.

To show Map is not full, we need an equivariant map $\text{Map}X \to \text{Map}Y$ not equal to $\text{Map}f$ for any $f : X \to Y$. When $X = Y$, we can construct an equivariant map this way: let $g \in G$, and define $u_g : k \mapsto u(gk)$, and finally $\hat{g} : u \mapsto u_g$. In other words, we act on the left. This is equivariant because

$$(h \cdot (\hat{g}u))(k) = (h \cdot u_g)(k) = u_g(kh) = u(gkh) = (h \cdot u)(gk) = (\hat{g}(h \cdot u))(k)$$

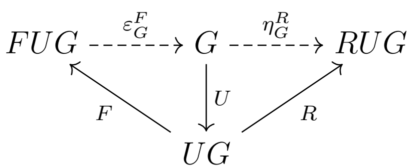
In other words, since $\hat{g}$ acts on the left and $h\cdot$ on the right, they don't interfere with each other and can be applied in either order with the same result.

Now we have to insure that $\hat{g} \neq \text{Map}f$ for all $f : X \to X$. Let X have two elements, say $X = \{0, 1\}$. Let u be the function taking $1 \in G$ to 1 and all other elements of G to 0. Then $u_g(k) = u(kg)$ equals 1 iff $k = g^{-1}$. So if $g_1 \neq g_2$, then $\widehat{g_1} \neq \widehat{g_2}$. So as g ranges over G, we get $|G|$ different equivariant functions $\hat{g}$. On the other hand, there are only four functions $f : X \to X$, and so only four different Map$f$'s. Thus if $|G| > 4$, Map is not full. (With a moment's thought you can see that this works whenever $|X| > 1$ and $|G| > |X^X|$.)

Monoids and groups: $F \dashv U \dashv R$. (See §2.2 for some background.)

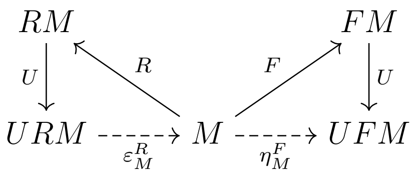
Example 2.1.3(d) (p.45) introduced the functor R : **Monoid** $\to$ **Group**, sending a monoid to its group of invertible elements. R is the right adjoint to U. I don't know a standard term for RM, but I'll call it the *invertible core* of the monoid. People write $M^\times$ for it (at least when '$\cdot$' is used for the operation of the monoid M), not distinguishing between RM and URM.

We construct the (by now familiar) diagram:



All elements of a group G are invertible, so η^R_G is the identity map on G. It turns out that ε^F_G is too: if a monoid M happens to be a group, then UFM is just M. As expected, U is full and faithful: if $f : M \to M'$ is a monoid homomorphism, and M and M' happen to be groups, then f is also a group homomorphism. The "groupness" of the source and target don't impose any additional constraints on f. (Contrast with the forgetful functor from **Group** to **Set**.)

Here's the other diagram:



Start with right hand triangle. F is neither full nor faithful, and η_M^F is neither split epic nor monic (in general). First example: $\eta_{\mathbb{N}} : \mathbb{N} \to \mathbb{Z}$, under addition. Although epic, $\eta_{\mathbb{N}}$ is not surjective (see Exericse 5.2.23, §5.12) and hence not split epic. F isn't full, because if $p : \mathbb{N} \to \mathbb{N}$ is a monoid morphism, then $Fp : \mathbb{Z} \to \mathbb{Z}$ must send 1 to a nonnegative integer, unlike some group morphisms from $\mathbb{Z}$ to $\mathbb{Z}$.

Neither is η_M always monic. The monoid of all functions from a two-element set to itself furnishes a counterexample (see §2.2), and shows that F is neither faithful nor full.

Turn to the left hand triangle. R isn't faithful, and ε_M^R isn't epic for certain monoids. Simple example: the only invertible element of $\mathbb{N}$ is 0, so $UR\mathbb{N} \to UR\mathbb{N}$ devolves to the trivial morphism from $\{0\}$ to $\{0\}$. But we have a morphism $\hat{k} : \mathbb{N} \to \mathbb{N}$, $\hat{k} : n \mapsto kn$ for every $k \in \mathbb{N}$. Obviously $\varepsilon_{\mathbb{N}}^R$ isn't surjective, but it's also not epic: $\varepsilon_{\mathbb{N}}^R \hat{k} = \varepsilon_{\mathbb{N}}^R$ for all k.

Note that ε_M^R is the inclusion morphism, so it's injective and hence monic, but is it split monic? That is, do we have a morphism $\varphi : M \to URM$ that is the identity when restricted to URM? Not in general. Simple counterexample: $M = (\mathbb{Z}, \cdot)$. $URM = \{\pm 1\}$. A morphism $\varphi : M \to URM$ must satisfy $(\varphi 0)(\varphi 0) = \varphi 0$, hence $\varphi 0 = 1$, hence $\varphi(-1) = \varphi(-1)1 = \varphi(-1)(\varphi 0) = \varphi(-1 \cdot 0) = \varphi 0 = 1$. But $-1 \in URM$, so φ restricted to URM isn't the identity. Variation: let $M = G \sqcup \{0\}$ where G is any nontrivial group and we define $0 \cdot g = g \cdot 0 = 0$ for all $g \in G$. Then $URM = G$ and the same argument shows that $\varphi : M \to G$ must send everything to 1.

Abelian Groups: $A \dashv U$. U is full and faithful because homomorphisms between two groups which both *happen to be abelian* are just group homomorphisms. (Just like the forgetful functor from **Group** to **Monoid**.) And the counit is an isomorphism.

The unit η_X is the canonical epimorphism of X onto the abelianized UAX. A is neither faithful nor full. Thus, η_X is neither monic nor split epic. Not monic: look at the exact sequences

$$C_2 \rightrightarrows S_3 \overset{\eta}{\twoheadrightarrow} C_2$$

There are three transpositions in S_3, and so three ways to inject C_2 into S_3. The commutator subgroup of S_3 is $A_3 = C_3$, so $A(S_3) = C_2$. Therefore the compositions are all the identity.

I found an example on the internet showing that A is not full. Let Q be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. The commutator subgroup is $\{\pm 1\}$ (easy to check, since -1 is in the center and i, j, k have cyclic symmetry). So $AQ = C_2 \times C_2$. There are only two homomorphisms from C_2 to Q (since -1 is the only element of order 2), but four homomorphisms from $A(C_2)$ to $A(Q)$, i.e., from C_2 to $C_2 \times C_2$.

Liberation: $F \dashv U$. The remaining two examples from figs.1 and 2 involve a free functor and a forgetful functor, in each case going between a category of "algebraic things" and **Set**. The unit $\eta_X : X \to UFX$ maps a set to the underlying set of the "free algebraic thing"; I call this **liberation** (instead of *freeification*). The counit $\varepsilon_A : FUA \to A$ corresponds to a kind of evaluation.

The figures give just two examples, **Vect** and **G-set**. Free groups and free monoids follow the same pattern. **G-set** by far provides the simplest illustration, and it's typical. (See §2.5 for definitions.)

Let X be a set; then $UFX = G \times X$ and $\eta_X : x \mapsto (1, x)$. We see immediately that η_X is monic ($\equiv$ injective) but not split epic ($\equiv$ surjective), provided $G \neq 1$. Thus F is faithful but not full. To see these facts explicitly, note that if $\varphi : X \to Y$ is a function, then $F\varphi : (g, x) \to (g, \varphi x)$.

So if $\varphi \neq \varphi'$ then clearly $F\varphi \neq F\varphi'$. To show non-fullness, let $X = Y$; pick a $g_0 \in G$, $g_0 \neq 1$, and define $\psi : (g, x) \mapsto (gg_0, x)$. Verification that ψ is equivariant takes but a moment, and obviously ψ is not $F\varphi$ for any $\varphi : X \to X$.

Let A be a G-set. Then FUA is the set $G \times A$ equipped with the action $g \cdot (h, a) \mapsto (gh, a)$, and $\varepsilon_A : (h, a) \mapsto h \cdot a$. We check that ε_A is equivariant:

$$\begin{array}{ccc} (h,a) & \xmapsto{\varepsilon_A} & h \cdot a \\ {\scriptstyle g\cdot}\downarrow & & \downarrow{\scriptstyle g\cdot} \\ (gh,a) & \xmapsto{\varepsilon_A} & gh \cdot a \end{array}$$

We see immediately that ε_A is surjective and thus epic, but isn't injective (for $G \neq 1$) and thus isn't split monic. Hence U is faithful but not full. Both these facts can be seen directly without breathing hard.

2.12 2.2.13(a), p.57

Write fA for the image of $A \subseteq K$, and $f^*B = f^{-1}B$ for the inverse image of $B \subseteq L$. Partition L into $I \sqcup N$ with $I = fK$ and $N = L \smallsetminus fK$. Also write $a \equiv a'$ if $f(a) = f(a')$, and let $[a]$ be the equivalence class of a. For $A \subseteq K$, let $\widehat{A}$ be the union of the equivalence classes of elements of A; in other words, $\widehat{A} = \{x | x \equiv a \text{ for some } a \in A\}$.

The following two computations are routine:

$$ff^*B = B \cap I \subseteq B$$
$$f^*fA = \widehat{A} \supseteq A$$

Recall that the categories $\mathcal{P}(K)$ and $\mathcal{P}(L)$ are posets, with a unique morphism $S \xrightarrow{\subseteq} T$ when and only when $S \subseteq T$. The equations just given show that we have a unit and a counit:

$$\eta_A : A \overset{\subseteq}{\Rightarrow} f^* f A$$
$$\varepsilon_B : f f^* B \overset{\subseteq}{\Rightarrow} B$$

The triangle identities are satisfied automatically, since f and f^* are functors (i.e., order-preserving) between posets. So by Theorem 2.2.5 (p.53) of Leinster, $f \dashv f^*$.

For the right adjoint to f^*, we need to find a $g : \mathcal{P}(K) \to \mathcal{P}(L)$ such that $B \subseteq g f^* B$ and $f^* g A \subseteq A$ for all $A \subseteq K$, $B \subseteq L$. For any $A \subseteq K$, define A° to be the union of those equivalence classes that are contained in A, i.e., $A^\circ = \bigcup_{[a] \subseteq A} [a]$. So $\widehat{A^\circ} = f^* f(A^\circ) = A^\circ$. Also, $(f^* B)^\circ = f^* B$.

Set

$$gA = f(A^\circ) \cup N$$

(Recall that N is the complement of the image fK, so $f^* N = \varnothing$. Also, the union is disjoint, but we won't need that.) We compute:

$$\begin{aligned} f^* g A &= f^*(f(A^\circ) \cup N) \\ &= f^* f A^\circ \cup f^* N = A^\circ \cup \varnothing \\ &= A^\circ \subseteq A \end{aligned}$$

and

$$\begin{aligned} g f^* B &= f((f^* B)^\circ) \cup N \\ &= f f^* B \cup N = (B \cap I) \cup N \\ &\supseteq (B \cap I) \cup (B \cap N) = B \end{aligned}$$

So $f^* \dashv g$.

2.13 2.2.13(b), p.57

Notation as in part (a). We write Sx to mean $x \in S$, and xRy to mean $(x, y) \in R$. Since p is onto, $N = \varnothing$ and $gR = pR^\circ$. So:

$$
\begin{aligned}
pR = S: &\quad Sx \Leftrightarrow (\exists y)xRy \\
p^*S = R: &\quad xRy \Leftrightarrow Sx \\
(x,y) \equiv (x',y'): &\quad x = x' \\
[(x,y)] &= \{x\} \times Y \\
R^\circ = \bigcup_{[(x,y)] \subseteq R} [(x,y)]: &\quad xR^\circ y \Leftrightarrow (\forall y')xRy' \\
gR = pR^\circ = S: &\quad Sx \Leftrightarrow (\exists y)xR^\circ y \Leftrightarrow (\exists y)(\forall y')xRy' \\
&\quad \Leftrightarrow (\forall y')xRy'
\end{aligned}
$$

We've made use of the fact that the assertion $(\exists y)\varphi(x)$ is equivalent to $\varphi(x)$ if the latter does not mention y.

Units and counits for $p \dashv p^*$:

$$
\begin{aligned}
x(p^*pR)y &\Leftrightarrow (pR)x \Leftrightarrow (\exists y)xRy \\
(pp^*S)x &\Leftrightarrow (\exists y)x(p^*S)y \Leftrightarrow (\exists y)Sx \Leftrightarrow Sx
\end{aligned}
$$

So:

$$
\begin{aligned}
R \subseteq p^*pR: &\quad xRy \Rightarrow (\exists y)xRy \\
pp^*S \subseteq S: &\quad Sx \Rightarrow Sx
\end{aligned}
$$

Units and counits for $p^* \dashv g$:

$$
\begin{aligned}
(gp^*S)x &\Leftrightarrow (\forall y')x(p^*S)y' \Leftrightarrow (\forall y')Sx \Leftrightarrow Sx \\
x(p^*gR)y &\Leftrightarrow (gR)x \Leftrightarrow (\forall y)xRy
\end{aligned}
$$

So:

$$
\begin{aligned}
S \subseteq gp^*S: &\quad Sx \Rightarrow Sx \\
p^*gR \subseteq R: &\quad (\forall y)xRy \Rightarrow xRy
\end{aligned}
$$

(Changing dummy variables may make these look a bit nicer: $xRy \Rightarrow (\exists y')xRy'$ and $(\forall y')xRy' \Rightarrow xRy$.)

2.14 2.2.14, p.57

This is a classic "unwind the definitions" exercise. Since $F \dashv G$, we have the unit/counit pair η/ε satisfying the triangle identities. We need to construct a unit/counit pair η^*/ε^* with $\eta^* : 1_{[\mathcal{A},\mathcal{S}]} \Rightarrow F^*G^*$, $\varepsilon^* : G^*F^* \Rightarrow 1_{[\mathcal{B},\mathcal{S}]}$, satisfying the triangle identities.

First we write out what objects, morphisms, functors, natural transformations, and units and counits look like for the functor categories $[\mathcal{A},\mathcal{S}]$ and $[\mathcal{B},\mathcal{S}]$. Rather than describe functors in general, we just consider F^* and G^*, and how they act on objects and morphisms. Among the natural transformations, we consider only the unit and counit. As usual, $f : A \to A'$ is a typical morphism between objects of $\mathcal{A}$, likewise $g : B \to B'$ for $\mathcal{B}$.

Objects:	$X : \mathcal{A} \to \mathcal{S}$	$Y : \mathcal{B} \to \mathcal{S}$
Morphisms:	$\alpha : X \Rightarrow X'$ $\alpha_A : XA \to X'A$	$\beta : Y \Rightarrow Y'$ $\beta_B : YB \to Y'B$
Functors:	$G^*X = XG : \mathcal{B} \to \mathcal{S}$ $G^*X : B \mapsto XGB$ $G^*X : g \mapsto XGg$ $G^*\alpha = \alpha G : XG \Rightarrow X'G$ $\alpha_{GB} : XGB \to X'GB$	$F^*Y = YF : \mathcal{A} \to \mathcal{S}$ $F^*Y : A \mapsto YFA$ $F^*Y : f \mapsto YFf$ $F^*\beta = \beta F : YF \Rightarrow Y'F$ $\beta_{FA} : YFA \to Y'FA$
Unit and counit:	$\eta^*_X : X \Rightarrow F^*G^*X = XGF$ $(\eta^*_X)_A : XA \to XGFA$	$\varepsilon^*_Y : G^*F^*Y = YFG \Rightarrow Y$ $(\varepsilon^*_Y)_B : YFGB \to YB$

The naturality squares for η^*_X and ε^*_Y look like this:

$$
\begin{array}{ccc}
XA & \xrightarrow{(\eta^*_X)_A} & XGFA \\
{\scriptstyle Xf}\downarrow & & \downarrow{\scriptstyle XGFf} \\
XA' & \xrightarrow[(\eta^*_X)_{A'}]{} & XGFA'
\end{array}
\qquad\qquad
\begin{array}{ccc}
YFGB & \xrightarrow{(\varepsilon^*_Y)_B} & YB \\
{\scriptstyle YFGg}\downarrow & & \downarrow{\scriptstyle Yg} \\
YFGB' & \xrightarrow[(\varepsilon^*_Y)_{B'}]{} & YB'
\end{array}
$$

Compare with the naturality squares for η and ε:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & GFA \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\
A' & \xrightarrow[\eta_{A'}]{} & GFA'
\end{array}
\qquad\qquad
\begin{array}{ccc}
FGB & \xrightarrow{\varepsilon_B} & B \\
{\scriptstyle FGg}\downarrow & & \downarrow{\scriptstyle g} \\
FGB' & \xrightarrow[\varepsilon_{B'}]{} & B'
\end{array}
$$

Clearly if we apply X to the square for η, and Y to the square for ε, we get the squares for η^*_X and ε^*_Y. So we set

$$(\eta^*_X)_A = X\eta_A, \qquad (\varepsilon^*_Y)_B = Y\varepsilon_B$$

Now we have to check the triangle identities. These involve both horizontal and vertical composition of natural transformations (pp.30, 37), so here are the formulas for those:

$$(\alpha\beta)_A = \alpha_A\beta_A, \quad (F\alpha)_A = F\alpha_A, \quad (\alpha F)_A = \alpha_{FA}$$

(We need only the special cases of horizontal composition.) We can rewrite the definitions of η^* and ε^*:

$$\eta^*_X = X\eta, \qquad \varepsilon^*_Y = Y\varepsilon$$

For the original η and ε, the triangle identities read (p.52)

$$\varepsilon_F \; F\eta = 1_F, \qquad G\varepsilon \; \eta G = 1_G$$

Here is one of the triangle diagrams for η and ε, written at the component level, and the corresponding diagram for η^* and ε^*:

$$\begin{array}{ccc} FA & \xrightarrow{F\eta_A} & FGFA \\ & {\scriptstyle 1_{FA}} \searrow & \downarrow {\scriptstyle \varepsilon_{FA}} \\ & & FA \end{array} \qquad\qquad \begin{array}{ccc} G^*X & \xrightarrow{G^*\eta^*_X} & G^*F^*G^*X \\ & {\scriptstyle 1_{G^*X}} \searrow & \downarrow {\scriptstyle \varepsilon^*_{G^*X}} \\ & & G^*X \end{array}$$

In the right hand diagram, the arrows all represent natural transformations. We check that it commutes:

$$\begin{aligned} (\varepsilon^*_{G^*X})(G^*\eta^*_X) &= ((G^*X)\varepsilon)(G^*(X\eta)) \\ &= (XG\varepsilon)(X\eta G) = X(G\varepsilon\;\eta G) \\ &= X(1_G) = 1_{XG} \end{aligned}$$

These equations are all between natural transformations. For example, $(\varepsilon^*_{G^*X})$ is a natural transformation whose components are $(\varepsilon^*_{G^*X})_B$ (with $B \in \mathcal{B}$). Rewritten at the component level, the computation is longer, but maybe easier to follow.

$$\begin{aligned} ((\varepsilon^*_{G^*X})(G^*\eta^*_X))_B &= (\varepsilon^*_{G^*X})_B(G^*\eta^*_X)_B \\ &= ((G^*X)\varepsilon_B)(\eta^*_X G)_B \\ &= (XG\varepsilon_B)(\eta^*_X)_{GB} \\ &= (X(G\varepsilon_B))(X(\eta_{GB})) \\ &= X(G\varepsilon_B\;\eta_{GB}) = X(1_{GG}) = 1_{XGB} \end{aligned}$$

The verification of the other triangle identity is similar.

2.15 Initial Objects in the Comma Category

Among all the comma categories, the slice and coslice categories lend themselves best to intuition. The category $(A \Rightarrow G)$ lies just a tweak away from the coslice category $A/\mathcal{A}$ (p.60). So we'll start with that.

An initial object $e : A \to A_0$ of $A/\mathcal{A}$ has the defining property that given any $f : A \to B$, there is a unique q completing the diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & A_0 \\ & {}_{f}\searrow & \downarrow{\scriptstyle q} \\ & & B \end{array}$$

Obviously $1_A : A \to A$ satisfies this condition. Any A_0 that is uniquely isomorphic to A does also.

The category $(A \Rightarrow G)$ lifts the object B and morphism q to an "index" category $\mathcal{B}$. Think of $B \in \mathcal{B}$ as a "name" for the object $GB \in \mathcal{A}$, and q in $\mathcal{B}$ as a "name" for Gq. Instead of starting with an $f : A \to B$ (both A and B in $\mathcal{A}$), we start with $f : A \to GB$. Instead of asking for a unique $q : A_0 \to B$ in $\mathcal{A}$, we ask for a unique $q : B_0 \to B$ in $\mathcal{B}$. The commuting triangle still lives in $\mathcal{A}$:

$$\begin{array}{ccc} A & \xrightarrow{e} & GB_0 \\ & {}_{f}\searrow & \downarrow{\scriptstyle Gq} \\ & & GB \end{array}$$

Setting $B_0 = FA$ and $e = \eta_A$, we have the situation of Lemma 2.3.5 (p.60): (FA, η_A) is the initial object of $(A \Rightarrow G)$, or speaking casually, $\eta_A : A \to GFA$ is.

This lifting to $\mathcal{B}$ imposes three constraints. First, the morphism $f : A \to GB$ must be to a named object in $\mathcal{A}$. Second, the completing morphism Gq must be named. Third, the name q must be unique (*not* the image Gq). Let's see how these play out in some examples. (Consult figs.1 and 2.)

Topological Spaces. Let's think about completing the triangle, i.e., finding $q : FA \to B$ such that $f = (Gq)\eta_A$. In all three cases, f determines q "set-theoretically". That is, if f is in **Set** (case $D \dashv U$) then $Uq = f$;

if f is in **Top** (other two cases) then Uf determines q. In the first two cases ($U \dashv I$ and $D \dashv U$), f and q are "basically the same" once we ignore topologies. In the remaining case, we need to do a bit more work: we're given $f : A \to DB$ (A a locally connected space, B a set), and $a \in A$, and we need to figure out where $q[a]$ lands (with $[a]$ being the component of a). Chasing the diagram yields $q[a] = f(a)$. The result doesn't depend on the choice of a because f is continuous and DB is discrete. Note the critical role of the "target constraint".

In the case $D \dashv U$, we have a last step: we need to remember the topology of UB, taking us from Uq to q. All we really need to check is continuity, but this follows from the discreteness of the source: $q : DA \to B$.

Monoids. Case $F \dashv U$: given a monoid A, a group B, and a monoid homomorphism $f : A \to B$, we want a unique group homomorphism $q : FA \to B$ for which $f(a) = q(\eta_A(a))$ for all $a \in A$. Not much to say, except that the (non-trivial) construction of FA is designed to insure this. Caveat: η_A might not be injective.

Case $U \dashv R$: given a group A, a monoid B, and a group homomorphism $f : A \to RB$, we want a unique monoid homomorphism $q : UA \to B$ for which $f(a) = q(\eta_A(a))$ for all $a \in A$. This time $\eta_A = 1_A$, so $q = f$, pretty much. To be picky about it, f is a group homomorphism and q is a monoid homomorphism, so really $Uf = q$, but U doesn't do anything to f except change its official domicile.

Abelian groups. This is practically the same story as $U \dashv R$ for monoids.

G-sets. Orb $\dashv$ Triv. This resembles $C \dashv D$ for locally connected spaces. Given a G-set A, a set B, and an equivariant map $A \to \mathrm{Triv}(B)$, we want a unique $q : \mathrm{Orb}(A) \to B$ making the triangle commute. The target constraint (that G acts trivially on $\mathrm{Triv}(B)$) forces entire orbits of A to map to single elements of B. That defines q uniquely.

Triv $\dashv$ Fix. This resembles $U \dashv R$ for monoids, with "fixed point" replacing "invertible element". Given a set A, a G-set B, and a function $f : A \to \mathrm{Fix}(B)$, we want a unique equivariant $q : \mathrm{Triv}(A) \to B$ making the triangle commute. Essentially $q = f$, more precisely $q = \mathrm{Triv}(f)$.

$F \dashv U$. Given a function $f : A \to UB$ with A a set and B a G-set, we want a unique $q : FA \to B$ making the triangle commute. This time the "morphism constraint" plays the decisive role. The unit $\eta_A : A \to UFA$ is an injection $a \mapsto (1, a)$. So we have a well-defined $\bar{f} : \{1\} \times A \to UB$ with $f(a) = \bar{f}(\eta_A(a))$ for all a. The commuting triangle says that Uq extends $\bar{f}$ to all of $G \times A$. Now, extending $\bar{f}$ isn't hard, but it's the equivariance of q that makes it unique. The target constraint doesn't narrow things down at all, since any set can be turned into a G-set.

Vector spaces; groups. Similar to $F \dashv U$ for G-sets. Again it's the morphism constraint that makes things work. For groups, the target constraint doesn't really constrain. For vector spaces over $\mathbb{R}$, the target constraint says that UB must look like $\mathbb{R}^n$ for some n. (Of course, we could replace $\mathbb{R}$ with any other field.)

Final note: Suppose $U : \mathcal{C} \to \mathbf{Set}$ is a forgetful functor, and A is a set. The comma category $(A \Rightarrow U)$ consists of pairs $(C, f : A \to UC)$. If A is a singleton, these are "pointed $\mathcal{C}$ objects". Familiar examples: pointed sets and pointed topological spaces. The category $(A \Rightarrow U)$ generalizes this to "$\mathcal{C}$ objects with A dropped in". I like to think of the pair $(C, f : A \to UC)$ as a "labeled" object C; f "labels" certain elements of C. Morphisms in $(A \Rightarrow U)$ are morphisms in $\mathcal{C}$ that "respect the labels", i.e., the morphism takes any element labeled a to an element also labeled a, for all $a \in A$. Note that the same element of C can carry multiple labels: f needn't be injective.

We'll make use of labeled groups in §6.13.

2.16 2.3.8, p.63

Lemma 2.2.4 (p.52) says:

$$\overline{g} = (Gg)\eta_A; \qquad \overline{f} = \varepsilon_B(Ff)$$

where f and g are morphisms in $\mathcal{A}$ and $\mathcal{B}$ respectively. (Since $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are the unique objects of the two categories, the conditions $f : A \to GB$ and $g : FA \to B$ hold automatically.) Rewrite:

$$Gg = \overline{g}\eta_A^{-1}; \qquad Ff = \varepsilon_B^{-1}\overline{f}$$

Applying G to the equation for Ff, and then using the equation for Gg with $g = \overline{f}$:

$$GFf = (G\varepsilon_B)^{-1}G\overline{f} = (G\varepsilon_B)^{-1}\overline{\overline{f}}\eta_A^{-1}$$

In other words,

$$GFf = (G\varepsilon_B)^{-1}f\eta_A^{-1}$$

If $f = 1_A$, we get

$$1_A = (G\varepsilon_B)^{-1}\eta_A^{-1} \text{ so } \eta_A = (G\varepsilon_B)^{-1}$$

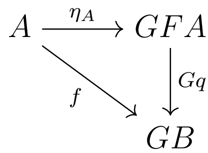
and so

$$GFf = \eta_A f \eta_A^{-1}$$

In other words, the group endomorphism $f \mapsto GFf$ is the same as the inner automorphism $f \mapsto \eta_A f \eta_A^{-1}$. Likewise, $FGg = \varepsilon_B^{-1} g \varepsilon_B$. It follows that F and G are group isomorphisms, and moreover F^{-1} is G composed with an inner automorphism.

2.17 2.3.10, p.63

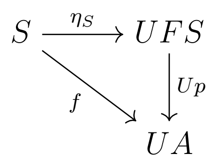
Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, with $\eta_A : A \to GFA$ and $\varepsilon_B : FGB \to B$ isomorphisms for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ (as in Def. 1.3.15 p.34). We have to show that for any $f : A \to GB \in \mathcal{A}$ there is a unique $q : FA \to B$ completing the diagram



Because η_A is an isomorphism, Gq is determined uniquely by $Gq = f\eta_A^{-1}$. Because G is an equivalence, it is full and faithful (Prop.1.3.18 p.36, proved in Ex.1.3.32 §1.8) so q is also determined uniquely.

2.18 2.3.11, p.63

Suppose UA has at least two elements. Let S be a set. Since η_S is an initial object of $(S \Rightarrow U)$ (Lemma 2.3.5, p.60), for any $f : S \to UA$ we can complete the diagram with a function Up:



If η_S isn't injective, say $\eta_S(x_1) = \eta_S(x_2)$ for $x_1 \neq x_2$, then any function $f : S \to UA$ with $f(x_1) \neq f(x_2)$ makes it impossible to complete the diagram.

Say FS is the free group for a set S. This exercise says that we can regard S as a subset of FS. (Technically, there is a canonical injection of S into FS, or even more technically, into UFS.) Contrast with the free group FM for a monoid M, where the map $M \to UFM$ isn't always injective. (For example, when cancellation fails in M, i.e., when $cx_1 = cx_2$ with $c \neq 0$ and $x_1 \neq x_2$, $M \to UFM$ isn't injective.)

2.19 2.3.12, p.64

For every A in **Par**, add a new element $*_A$ to it to get a set in $\mathbf{Set}_*$. So we have a mapping $FA = A \sqcup \{*_A\}$ from **Par** to $\mathbf{Set}_*$. To make F a functor, let f be a partial function $f : A \dashrightarrow B$, say $f : S \to B$ with $S \subseteq A$. Define Ff by having it agree with f on S and having it send everything in $A \smallsetminus S$ to $*_B$. Also of course Ff sends $*_A$ to $*_B$. For the reverse functor G from $\mathbf{Set}_*$ to **Par**, let GA be A with the basepoint removed. If $g : A \to B$ is a basepoint preserving function, let Gg be the partial function obtained by first restricting g to GA, and then letting $Gg(a)$ be undefined for all $a \in GA$ for which $g(a)$ is the basepoint of B. It is routine to verify that FG and GF are equivalent to the identities on $\mathbf{Set}_*$ and **Par**.

As for the choice of $*_A$, by far the simplest solution is to let $*_A = \{A\}$, since the Foundation axiom of set theory forbids a set from being an element of itself.

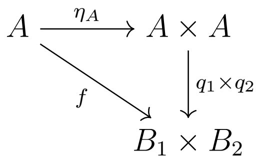
To interpret $\mathbf{Set}_*$ as a coslice category, let $*$ be a singleton (say, $\{\varnothing\}$). Objects of $*/\mathbf{Set}$ are functions $f : * \to A$, or basically nonempty sets with basepoints. Writing A_* for $f : * \to A$ and B_* for $g : * \to B$, a morphism $A_* \to B_*$ is a function $h : A \to B$ such that the composition $* \xrightarrow{f} A \xrightarrow{h} B$ is $* \xrightarrow{g} B$, i.e., a basepoint preserving function.

3 Interlude on Sets

3.1 3.1.1, p.73

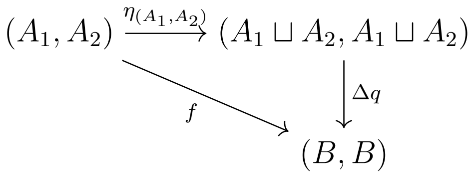
The right adjoint is the cross-product $(A, B) \mapsto A \times B$, the left adjoint is the disjoint union $(A, B) \mapsto A \sqcup B$. The initial object property (Theorem 2.3.6, p.61) provides the easiest demonstrations. First, the cross-product: the unit η_A is $a \mapsto (a, a)$. We have to verify that for any $f : A \to B_1 \times B_2$

we have a unique $(q_1, q_2) : (A, A) \to (B_1, B_2)$ with

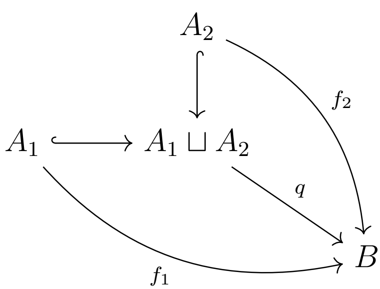


However, a function $f : A \to B_1 \times B_2$ is basically a pair of functions $f_1 : A \to B_1$, $f_2 : A \to B_2$, so obviously $(q_1, q_2) = (f_1, f_2)$ is the unique completing morphism.

Next, disjoint union. The unit maps $(A_1, A_2) \to (A_1 \sqcup A_2, A_1 \sqcup A_2)$, and sends $a_1 \in A_1$ to (a_1, a_1), $a_2 \in A_2$ to (a_2, a_2). (Here we regard A_1 and A_2 as subsets of $A_1 \sqcup A_2$, a harmless abuse of notation.) We want the unique $\Delta q = (q, q)$ completing the diagram



Now $f : (A_1, A_2) \to (B, B)$ is just a pair of functions $f_1 : A_1 \to B$, $f_2 : A_2 \to B$. There is a unique q completing the diagram

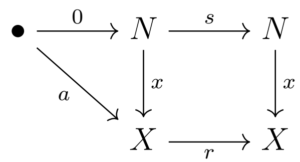


defined by $q(a_1) = f_1(a_1)$ for $a_1 \in A_1$, $q(a_2) = f_2(a_2)$ for $a_2 \in A_2$. Then Δq is the completing morphism for the previous diagram.

Leinster introduces products and coproducts in Chapter 5, at which point we'll see that the cross-product is the product and the disjoint union is the coproduct in **Set**.

3.2 3.1.2, p.73

Let the objects of $\mathcal{C}$ be triples $(N, 0, s)$ where X is a set, $0 : \bullet \to N$ is a function from some arbitary fixed singleton set $\bullet$ to N, and $s : N \to N$ is a function from N into itself. The morphisms of $\mathcal{C}$ are commuting diagrams



$(\mathbb{N}, 0, s)$ is an initial object in this category. (Note that the two functions labelled x in the diagram must be the same, i.e., this is part of the definition of the morphisms. We regard 0 as a function singling out the element zero of $\mathbb{N}$, likewise a.)

3.3 3.2.14, p.77

(a) Assume $F \dashv U$. Let $S = \mathcal{P}\left(\bigcup_{i \in I} UA_i\right)$. Then $|UA_i| < |S|$ for all i. Let $A = FS$. If $A \cong A_i$, then $UA \cong UA_i$, i.e., $|UA| = |UA_i|$. But we know that

$$|UA_i| < |S| \leq |UFS| = |UA|$$

with $|S| \leq |UFS|$ because $\eta_S : S \to UFS$ is injective (Ex. 2.3.11 §2.18).

(b) This is trivial: pick a representative A_i from each equivalence class. If $\mathcal{A}$ were essentially small, the A_i's would form a small family $(A_i)_{i \in I}$ with I a set. By (a), there is an $A \in \mathcal{A}$ not isomorphic to any of the A_i's.

(c) Also trivial, since all these categories satisfy the hypotheses of (a).

3.4 3.2.16, p.78

This resembles the chain of adjoints for locally connected topological spaces. For a set S we define the indiscrete small category IS as having all the elements of S as objects, and a unique morphism from a to b for any $a, b \in S$. (So IS is a preorder.) The discrete small category DS has all the elements of S as objects, and no morphisms except for the identities. Given any small category T, there are unique functors $DS \to T$ and $T \to IS$.

Note that if $F : T \to DS$ is a functor, and if there is a morphism $a \to b$ between two objects of T, then we must have $Fa = Fb$. This inspires the definition of CT for a small category T: form the graph whose nodes are the objects of T, with an undirected edge between any two nodes when a morphism exists between them. Let CT be the set of connected components of this graph. It is routine to check that C is left adjoint to D.

For OT the set of objects of T, the adjunctions $D \dashv O \dashv I$ are immediate.

4 Representables

4.1 4.1.27, p.93

The isomorphism $H_A \cong H_{A'}$ means that for any $B \in \mathcal{A}$ we have an isomorphism in **Set** $\eta_B : H_A(B) \to H_{A'}(B)$, i.e., a bijection between $\mathbf{Set}(B, A)$ and $\mathbf{Set}(B, A')$. Borrowing notation from adjoints, let's write $\overline{f} : B \to A'$ for the function corresponding to $f : B \to A$, and also $\overline{g} : B \to A$ for the function corresponding to $g : B \to A'$. We have $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$.

The naturality requirement says that for any $q : B \to B'$, this diagram commutes:

$$\begin{array}{ccc} H_A(B) & \xleftarrow{H_A(q)} & H_A(B') \\ \eta_B \downarrow & & \downarrow \eta_{B'} \\ H_{A'}(B) & \xleftarrow[H_{A'}(q)]{} & H_{A'}(B') \end{array}$$

Now, $H_A(q)$ and $H_{A'}(q)$ are both pullbacks, where we compose with q on the right: $H_A(q) : f \mapsto fq$, likewise for $H_{A'}(q)$. So naturality says that:

$$\overline{fq} = \overline{f}q \text{ for any } q : B \to B' \text{ and any } f : B' \to A'$$
$$\overline{gq} = \overline{g}q \text{ for any } q : B \to B' \text{ and any } g : B' \to A$$

We apply the first equation with $B = A$, $B' = A'$, and $f = 1_{A'}$; we apply the second with $B = A'$, $B' = A$, and $g = 1_A$. This gives us:

$$\overline{q} = \overline{1_{A'}}\,q \text{ for any } q : A \to A'$$
$$\overline{p} = \overline{1_A}\,p \text{ for any } p : A' \to A$$

where we've replaced q with p in the second equation for clarity, since the domains and codomains are switched.

Note that $\overline{1_{A'}} : A' \to A$ and $\overline{1_A} : A \to A'$. So we can set $q = \overline{1_A}$ and $p = \overline{1_{A'}}$. This gives us:

$$1_A = pq$$
$$1_{A'} = qp$$

with $q : A' \to A$ and $p = A \to A'$. In other words, $A \cong A'$.

4.2 4.1.28, p.93

For G a group and p a prime, $U_p(G) = \{x \in G | x^p = 1\}$. If $h : G \to H$ is a group homomorphism, then h maps $U_p(G)$ into $U_p(H)$ since $[h(x)]^p = h(x^p) = h(1) = 1$. So we let $U_p(h)$ be the restriction of h to $U_p(G)$.

As Leinster noted on p.83, there is a unique homomorphism $\mu_x : \mathbb{Z}_p \to G$ for any $x \in U_p(G)$ determined by setting $\mu_x(1) = x$ (and hence $\mu_x(n) = x^n$); conversely, $\varphi(1) \in U_p(G)$ for any homomorphism $\varphi : \mathbb{Z}_p \to G$. (Note: $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for us, not the p-adics.) So we have the required bijection between $U_p(G)$ and $\mathbf{Group}(\mathbb{Z}_p, G)$, namely $x \leftrightarrow \mu_x$. Naturality is expressed by the commutative diagram

$$\begin{array}{ccc} x & \longleftrightarrow & \mu_x \\ {\scriptstyle U_p(h)} \big\downarrow & & \big\downarrow {\scriptstyle H^{\mathbb{Z}_p}(h)} \\ h(x) & \longleftrightarrow & h\mu_x \end{array}$$

or by the equation (for any $h : G \to H$)

$$\mu_{h(x)} = h\mu_x$$

But for any $n \in \mathbb{Z}$, we have

$$\mu_{h(x)}(n) = (h(x))^n; \qquad h\mu_x(n) = h(x^n)$$

so naturality holds. (It would have been enough to check the equation for $n = 1$, getting $h(x)$ on both sides.)

4.3 4.1.29, p.93

This is virtually the same as Exercise 4.1.28. Exercise 0.13(a) says that for every commutative ring R and every element $r \in R$, there is a unique ring homomorphism $\mu_r : \mathbb{Z}[x] \to R$ defined by $\mu_r(x) = r$; hence, every ring homomorphism $\varphi : \mathbb{Z}[x] \to R$ is μ_r for $r = \varphi(x)$. (Recall that every commutative ring is assumed to have a multiplicative identity, and every ring homomorphisms is assumed to preserve it: p.2.)

So again we have a 1–1 correspondence $r \leftrightarrow \mu_r$, and to check naturality, we check the equation

$$\mu_{h(r)} = h\mu_r$$

which we check by evaluating at $x \in \mathbb{Z}[x]$, getting $h(r)$ on both sides.

4.4 4.1.30, p.93

Let $S = \{0, 1\}$ with $\{1\}$ open and $\{0\}$ closed. (I think this reverses the usual convention, but it's more convenient for this problem.) Let $f : X \to S$ be continuous; then $f^{-1}\{1\}$ is open. Conversely, if U is an open subset of X, then the characteristic function χ_U is continuous. So there is a bijection between $\mathbf{Top}(X, S)$ and open subsets of X, $U \leftrightarrow \chi_U$.

To show that H_S is equivalent to $\mathcal{O}$, we need to check naturality. Let $h : X \to Y$ be continuous. Here we are dealing with contravariant functors, so naturality boils down to the equation

$$\chi_{h^{-1}(V)} = \chi_V h \qquad (V \subseteq Y);$$

note that the left hand side is $\chi_{\mathcal{O}(h)(V)}$, and the right hand side is $H_S(h)(\chi_V)$. We verify that the two functions (both mapping $X \to S$) are the same by showing they both give the inverse image of $\{1\}$. This follows from these equivalences, for any $x \in X$:

$$\chi_{h^{-1}(V)}(x) = 1 \Leftrightarrow x \in h^{-1}(V) \Leftrightarrow h(x) \in V \Leftrightarrow \chi_V(h(x)) = 1$$

4.5 4.1.31, p.93

Let $\mathcal{T}$ be the "arrow category" consisting of two objects 0 and 1 with three morphisms: 1_0, 1_1, and a unique morphism $a : 0 \to 1$. We show that M is naturally equivalent to $H^{\mathcal{T}} = \mathbf{Cat}(\mathcal{T}, -)$.

For any small category $\mathcal{A}$, an element of $\mathbf{Cat}(\mathcal{T}, \mathcal{A})$ is a functor $F : \mathcal{T} \to \mathcal{A}$. So it determines objects $F0, F1 \in \mathcal{A}$, and a morphism $Fa : F0 \to F1$. (The identity morphisms 1_0 and 1_1 take care of themselves.) Conversely, given any morphism $f : A_0 \to A_1$ in $\mathcal{A}$, we construct a functor F by setting $F0 = A_0$, $F1 = A_1$, and $Fa = f$. This gives us the required 1–1 mapping $\alpha_{\mathcal{A}} : H^{\mathcal{T}}(\mathcal{A}) \to M(\mathcal{A})$. The naturality of α is easily checked.

4.6 4.1.32, p.93

Suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$. By definition, F is left-adjoint to G iff for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have a 1–1 correspondence between $\mathcal{B}(FA, B)$ and $\mathcal{A}(A, GB)$ (denoted by $^{-}$ in both directions) satisfying

$$\begin{aligned} \overline{qg} &= (Gq)\overline{g}, \quad (g : FA \to B,\, q : B \to B') \\ \overline{fp} &= \overline{f}(Fp), \quad (f : A \to GB, p : A' \to A) \end{aligned}$$

(see pp.41–42, eqs.(4.2.1)–(4.2.3)).

By definition, the functors $\mathcal{B}(F(-), -)$ and $\mathcal{A}(-, G(-))$ are naturally isomorphic iff for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have a 1–1 correspondence $\alpha_{AB} : \mathcal{B}(FA, B) \to \mathcal{A}(A, GB)$ satisfying certain naturality requirements. By Exercise 1.3.29 (p.39), these requirements are equivalent to two commutative diagrams (with p and q as above):

$$\begin{array}{ccc} \mathcal{B}(FA,B) & \xrightarrow{q\circ -} & \mathcal{B}(FA,B') \\ {\scriptstyle \alpha_{AB}}\downarrow & & \downarrow{\scriptstyle \alpha_{AB'}} \\ \mathcal{A}(A,GB) & \xrightarrow[Gq\circ -]{} & \mathcal{A}(A,GB') \end{array} \qquad \begin{array}{ccc} \mathcal{B}(FA,B) & \xrightarrow{-\circ Fp} & \mathcal{B}(FA',B) \\ {\scriptstyle \alpha_{AB}}\downarrow & & \downarrow{\scriptstyle \alpha_{A'B}} \\ \mathcal{A}(A,GB) & \xrightarrow[-\circ p]{} & \mathcal{A}(A',GB) \end{array}$$

Let's say we write $\alpha_{AB}(f) = \overline{f}$ for $f : A \to GB$, $\alpha_{AB}^{-1}(g) = \overline{g}$ for $g : FA \to B$. The diagrams become:

$$\begin{array}{ccc} g & \xmapsto{q\circ -} & qg \\ \downarrow & & \downarrow \\ \overline{g} & \xmapsto[Gq\circ -]{} & (Gq)\overline{g} = \overline{qg} \end{array} \qquad \begin{array}{ccc} \overline{f} & \xrightarrow{-\circ Fp} & \overline{f}(Fp) = \overline{fp} \\ \uparrow & & \uparrow \\ f & \xrightarrow[-\circ p]{} & fp \end{array}$$

exactly the same as the naturality equations for adjoints.

4.7 4.2.2, p.99

The cornerstone of the Yoneda lemma is the diagram on p.97, for $f : B \to A$:

$$\begin{array}{ccc} \mathcal{A}(A,A) & \xrightarrow{H_A(f) = -\circ f} & \mathcal{A}(B,A) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ XA & \xrightarrow[Xf]{} & XB \end{array}$$

evaluated at $1_A \in \mathcal{A}(A, A)$:

$$\begin{array}{ccc} 1_A & \overset{H_A(f)}{\longmapsto} & f \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ \alpha_A(1_A) & \underset{Xf}{\longmapsto} & \begin{array}{c}\alpha_B(f) = \\ Xf(\alpha_A(1_A))\end{array} \end{array}$$

giving eq.(4.5): $Xf(\alpha_A(1_A)) = \alpha_B(f)$. This tells us that the natural transforation α is determined completely by the value of α_A at 1_A.

Reversing the arrows, we get almost the exact same diagrams. Only changes: $f : A \to B$ instead of $f : B \to A$; $H^A(f)$ instead of $H_A(f)$; $\mathcal{A}(A, B)$ instead of $\mathcal{A}(B, A)$. So we have

$$[\mathcal{A}, \mathbf{Set}](H^A, X) \cong X(A)$$

instead of

$$[\mathcal{A}^{\text{op}}, \mathbf{Set}](H_A, X) \cong X(A)$$

4.8 4.2.3, p.99

Let's write $\mathcal{M}$ for the "categorified" M, i.e., $\mathcal{M}$ has a unique object $*$, and the morphisms $* \to *$ of $\mathcal{M}$ are the elements of M.

(a) $H_*(*)$ is the set of all morphisms $* \to *$, i.e., the set of all elements of M. If m is a morphism, then $H_*(m)$ maps morphisms to morphisms by pre-composition: $x \mapsto xm$. But that's exactly the definition of $\underline{M}$.

(b) Let $\hat{X}$ be the functor corresponding to the right M-set X. So $\hat{X}(*) = X$, and $\hat{X}m : X \to X$ is defined by $x \mapsto x \cdot m$. Now let x be a fixed element of X, and define $\alpha_x : \underline{M} \to X$ by

$$\alpha_x p = x \cdot p \text{ for all } p \in M$$

We have $\alpha_x 1 = x\cdot 1 = x$, as required. An M-set map (i.e., a right equivariant map) must satisfy

$$(\alpha_x p) \cdot m = \alpha_x(p \cdot m) \text{ for all } m, p \in M$$

But this just says that

$$(x \cdot p) \cdot m = x \cdot (p \cdot m) \text{ for all } m, p \in M$$

and since $p \cdot m = pm$ by definition of $\underline{M}$, this becomes $(x \cdot p) \cdot m = x \cdot pm$, the definition of a right equivariant action. (If you want to get persnickety, in $p \cdot m$, we regard p as an element of $\underline{M}$ and m as an element of M; in pm, both p and m are elements of M.)

To show α_x is uniquely determined by x, suppose $\alpha : \underline{M} \to X$ is a right equivariant map, and let $x = \alpha 1$. Then for any $p \in M$, equivariance says that

$$\alpha p = \alpha(1 \cdot p) = (\alpha 1) \cdot p = x \cdot p$$

so $\alpha = \alpha_x$.

(c) The Yoneda lemma says that $\hat{X}(*) \cong [\mathcal{M}^{\mathrm{op}}, \mathbf{Set}](H_*, \hat{X})$. On the left hand side we have all the elements of X (i.e., the underlying set of X), and on the right hand side we have all the $\alpha : H_* \Rightarrow \hat{X}$. As we saw in (a), H_* is essentially the same as $\underline{M}$, and the α's are basically just right equivariant maps from $\underline{M}$ to X. In (b) we saw that the two sides are in 1–1 correspondence. The correspondence is natural in X because if $h : X \to X'$ is a right equivariant map, then $h\alpha : \underline{M} \to X'$ is also a right equivariant map, with corresponding element $(h\alpha)1 = h(\alpha 1)$. So the Yoneda lemma is proven for this case.

(d) Although the exercise didn't ask for it, let's look at the Yoneda embedding. For any $x \in M$, we have a corresponding $\alpha_x : \underline{M} \to \underline{M}$ defined by

$$\alpha_x p = x \cdot p \text{ for all } p \in M$$

Now let's compose the natural transformation α_x and α_y for $x, y \in M$, remembering that $a \cdot b = ab$ for all $a, b \in M$, i.e., the right M-action is the same as monoid multiplication.

$$\alpha_x(\alpha_y(p)) = \alpha_x(yp) = x(yp) = (xy)p = \alpha_{xy}p$$

So $\alpha_x\alpha_y = \alpha_{xy}$. Note that $x \mapsto \alpha_x$ is just the Yoneda mapping $H_\bullet$; we've just shown that it preserves the category structure.

(e) Finally we look at the "universal element" corollary, 4.3.2 (p.99). Say u is a universal element, i.e., $u \in \hat{X}(*) = X$ satisfying (4.6) (p.99):

$$(\forall B \in \mathcal{M})(\forall x \in \hat{X}B)(\exists!\, \bar{x} : B \to A)(\hat{X}\bar{x})u = x$$

Since $*$ is the only object of $\mathcal{M}$ and the morphisms $* \to *$ are just the elements of M, this becomes:

$$(\forall x \in X)(\exists!\, m \in M)(\hat{X}m)u = x$$

But $\hat{X}$ just sends $m \in M$ to the right action of m on X, so

$$(\forall x \in X)(\exists!\, m \in M)u \cdot m = x$$

This says that the map $m \mapsto u \cdot m$ is a bijection from M to X. But that map is α_u. So: if $\alpha : \underline{M} \Rightarrow X$ is a representation of $\overline{M}$, then we know from (b) that $\alpha = \alpha_u$ for some $u \in X$, and α_u must be a bijection by the definition of representation. Thus the universal element property (4.6) holds. In the other direction, if there is a u satisfying (4.6), then α_u is a bijection, and we already know it's an equivariant map, so it's a representation.

4.9 4.3.15, p.106

This is entirely straightforward. Being a bit sloppy, if f is an isomorphism in $\mathcal{A}$, then for some h, $fh = hf = 1$ so $J(f)J(h) = J(h)J(f) = 1$. (Really we should be fussier about the four different identities in those equations, but you get the idea.) If g is an isomorphism in $\mathcal{B}$, then for some k, $gk = kg = 1$; since J is full, g and k have preimages in $\mathcal{A}$, call them f and h, so $J(fh) = gk = 1$ and $J(hf) = kg = 1$; but since J is faithful, it follows that $fh = hf = 1$. Finally, (c) is an immediate consequence of (a) and (b).

4.10 4.3.16, p.106

(a) Suppose $f, g : A \to A'$ with $f \neq g$; we have to show that $H_\bullet(f) \neq H_\bullet(g)$. Now $H_\bullet(f) = H_f$, and H_f is the natural transformation that for any B maps $H_A(B) \to H_{A'}(B)$ via $H_f : p \mapsto fp$; likewise for $H_\bullet(g)$. Letting $B = A$ and $p = 1_A$, we have $H_f(1_A) = f \neq g = H_g(1_A)$. So $H_\bullet(f) \neq H_\bullet(g)$.

(b) If $\alpha : H_A \Rightarrow H_{A'}$, we have to show that $\alpha = H_\bullet(f)$ for some $f : A \to A'$. Basically we copy part of the proof of the Yoneda lemma. Set $f = \alpha_A(1_A)$, so $f : A \to A'$. For any $g \in H_A(B)$, we have to show that $\alpha_B(g) = (H_f)_B(g) = fg$. This follows from the diagrams

$$\begin{array}{ccc} H_A(A) & \xrightarrow{H_A(g)} & H_A(B) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ H_{A'}(A) & \xrightarrow[H_{A'}(g)]{} & H_{A'}(B) \end{array} \qquad \begin{array}{ccc} 1_A & \longmapsto^{H_A(g)} & g \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ f & \longmapsto_{H_{A'}(g)} & fg = \alpha_B(g) \end{array}$$

(c) Assuming the existence of a universal element $u \in XA$, we have to show that $H_A \cong X$, i.e., that there is a natural transformation $\alpha : H_A \Rightarrow X$ where $\alpha_B : H_A(B) \to XB$ is bijective for every $B \in \mathcal{A}$. We define the natural transformation $\tilde{u}$ by:

$$\tilde{u}_B : H_A(B) \to XB, \qquad \tilde{u}_B : f \mapsto Xf(u)$$

For any $B \in \mathcal{A}$, the universal element property (4.6, p.99) says

$$(\forall x \in XB)(\exists!\ f : B \to A) Xf(u) = x$$

or using $Xf(u) = \tilde{u}_B(f)$,

$$(\forall x \in XB)(\exists!\, f\!:\!B \to A)\tilde{u}_B(f) = x$$

This says that the map $\tilde{u}_B : H_A(B) \to XB$ is bijective, so we can take $\alpha = \tilde{u}$.

4.11 4.3.17, p.106

Let $\mathcal{D}$ be a discrete category. Note that for $\mathcal{D}$, contravariant and covariant means the same thing! Say $X : \mathcal{D} \to \mathbf{Set}$ is a functor; then X simply assigns a set XA to each $A \in \mathcal{D}$.

Next, let $A \in \mathcal{D}$ and look at $H^A = H_A$: this is defined by

$$H^A(B) = H_A(B) = \begin{cases} \{1_A\} \text{ if } A = B \\ \varnothing \text{ if } A \neq B \end{cases}$$

We might almost say that H_A is the characteristic function of A. Since all singletons are isomorphic in **Set**, we could pick some fixed singleton 1 to use for all $A \in \mathcal{D}$; then if $\chi_A(A) = 1$ and $\chi_A(B) = \varnothing$ for all $B \neq A$, we have $H_A \cong \chi_A$. I won't use the χ notation below, but it helps to "forget" that $H_A(A) = \{1_A\}$—all that matters is that $H_A(A)$ is a singleton, for which I'll write 1.

Next look at natural transformations $\alpha : H_A \Rightarrow X$. For all $B \neq A$, α_B must be the empty function to XB, so we can put those out of our minds. As for $\alpha_A : 1 \to XA$, this just amounts to an element of XA. So the Yoneda lemma is nearly trivial.

Now we look at the family $\{H_A | A \in \mathcal{D}\}$. If $A \neq B$, then we cannot have a natural transformation $\alpha : H_A \Rightarrow H_B$, since α_A would be a function $1 \to \varnothing$. On the other hand, there is obviously exactly one natural transformation from H_A to itself. So $\{H_A | A \in \mathcal{D}\}$ is a discrete subcategory of **Set**, equivalent to $\mathcal{D}$. In other words, the map $H_\bullet$ is an embedding.

Finally we look at the universal element corollary, 4.3.2 (p.99). We already know what $\alpha : H_A \Rightarrow X$ looks like. For this to be a representation, α must be an equivalence, i.e., XA must be a singleton, and all XB with $B \neq A$ must be empty. Consider condition (4.6) (p.99) on the universal element u:

$$(\forall B \in \mathcal{A})(\forall x \in XB)(\exists!\, \bar{x} : B \to A)(X\bar{x})u = x$$

For $B \neq A$, there can be no $\bar{x} : B \to A$, so XB must be empty (and making $XB = \varnothing$ satisfies the condition for B). For $B = A$, the only $\bar{x} : A \to A$ is 1_A, and $X1_A = 1_{XA}$, so we get:

$$(\forall x \in XA)u = x$$

i.e., XA must be a singleton.

4.12 Function Presheaves; Poset Categories

For some reason, Leinster omits one important special case from his Yoneda examples: the presheaf of continuous functions on a topological space T. It doesn't really matter what the target space is; to be concrete, let's say real-valued functions. Let $\mathcal{O}$ be the category of open subsets of X, with a morphism i_{UV} for every pair of open sets $U \subseteq V$. It's helpful to think of i_{UV} as the inclusion map. The contravariant functor X is defined by

$$\begin{aligned} XU &= \text{continuous functions on } U \\ Xi_{UV} &= \text{restriction from } V \text{ to } U \end{aligned}$$

Observe that restriction is actually composition with the inclusion map: $f|U = fi_{UV}$.

The topology and continuity don't actually play a role: it's enough if we have a family $\mathcal{O}$ of subsets of a set T, and for any $U \in \mathcal{O}$ a family XU of functions with domain U, such that restrictions work right: i.e., if $U \subseteq V$ and $U, V \in \mathcal{O}$ and $f \in XV$, then $f|U \in XU$. We then turn $\mathcal{O}$ into a category by introducing the inclusion maps as morphisms, and make X into a contravariant functor as before.

Next generalization: assume only that $\mathcal{O}$ is a poset, so that there is at most one morphism $i_{UV} : U \to V$ for any two objects U and V; we *define* $U \subseteq V$ to mean that i_{UV} exists. (Usually, we use $\leq$ instead of $\subseteq$ for this relation. But I will stick with $\subseteq$, and continue to refer to Xi_{UV} as a "restriction map", for evocativeness.) Since $\mathcal{O}$ is a poset, $U \subseteq V$ and $V \subseteq U$ imply that $U = V$, i.e., $\mathcal{O}$ is a skeletal category.

We now recognize that discrete categories (Ex.4.3.17, §4.11) constitute a special case of this. Also, monoids (Ex.4.2.3, §4.8) represent, so to speak, the obverse: there we have one object with many morphisms, instead of many objects with at most one morphism between each pair of objects.

We start with H_A and H^A. As in the discrete category case, we will replace all singletons in **Set** with a fixed one, denoted 1. So:

$$H_A(U) = \begin{cases} 1 \text{ if } U \subseteq A \\ \varnothing \text{ otherwise} \end{cases}$$

$$H^A(U) = \begin{cases} 1 \text{ if } A \subseteq U \\ \varnothing \text{ otherwise} \end{cases}$$

Next we look at natural transformations $\alpha : H_A \Rightarrow X$. Empty functions $\varnothing \to XU$ pose no problems, so we confine our attention to $1 \to XU$ for $U \subseteq A$. We see that α amounts to chosing an element $\alpha_U 1 \in XU$ for every $U \subseteq A$. Naturality demands, for $U \subseteq A$,

$$Xi_{UA}(\alpha_A 1) = \alpha_U 1$$

in other words, the chosen element $\alpha_U 1$ is the "restriction" of $\alpha_A 1$ to U. So to determine α, we pick an element of XA and then all other α_U fall out from that. That proves the Yoneda lemma.

It's a similar story for natural transformations $\alpha : H^A \Rightarrow X$. But this time X is a covariant functor, so if $A \subseteq U$, then Xi_{AU} must "extend" elements of XA to elements of XU. See the footnote[4] for an illustration. For the rest of this problem, I'll look only at the contravariant case.

Next, the Yoneda embedding. When do we have a natural transformation $H_A \Rightarrow H_B$? Precisely when $H_B(A) \neq \varnothing$, since as we've just seen, $H_A \Rightarrow X$ amounts to an element of XA. So $H_A \Rightarrow H_B$ exists when and only when $A \subseteq B$, in which case the natural transformation is unique.

Finally, the universal element corollary (4.3.2, p.99). For $\alpha : H_A \Rightarrow X$ to be a representation, XU must be a singleton for all $U \subseteq A$ and XU must be empty for all $U \not\subseteq A$. The universal element condition, on u:

$$(\forall U \in \mathcal{O})(\forall x \in XU)(\exists!\, \bar{x} : U \to A)(X\bar{x})u = x$$

Now, $(\exists!\, \bar{x} : U \to A)$ is equivalent to $U \subseteq A$. So for $U \not\subseteq A$, $XU = \varnothing$. For $U \subseteq A$, we have $\bar{x} = i_{UA}$ so $(X\bar{x})u = x$ becomes $x = Xi_{UA}(u)$, i.e., x is the restriction of u to U. Since this is true for all $x \in XU$, XU must be a singleton.

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[4]Let $\mathcal{F}_0$ be the category of fields of characteristic 0, with the morphisms being inclusions. Let the functor P take $K \in \mathcal{F}_0$ to the set of all polynomial functions on K, i.e., all functions given by polynomials in $K[x]$. Characteristic zero insures that the coefficients of a polynomial are determined uniquely by its associated function. If $K \subseteq L$, then any polynomial function on K obviously has a unique extension to L, namely the function with the same coefficients. Note that restricting a function in PL won't always give you a function in PK, since the former maps $L \to L$, the latter $K \to K$. (Or look at where the coefficients lie.)

It's a little odd to make the morphisms of $\mathcal{F}_0$ just inclusions. A field homomorphism is always a monomorphism (in fact injective), so usually we'd allow all imbeddings of one field into another. But then $\mathcal{F}_0$ wouldn't be a poset, or even a preorder (because of non-trivial automorphisms). Otherwise the example is unaffected: we can always extend uniquely, but not always restrict.

4.13 4.3.18, p.106

(a) First we unwind what full and faithful means for maps between functor categories. The objects of $[\mathcal{B}, \mathcal{C}]$ are functors $F : \mathcal{B} \to \mathcal{C}$ and the morphisms are natural transformations $\alpha : F \Rightarrow G$. Let's write $\hat{J}$ for $J \circ -$, so $\hat{J} : [\mathcal{B}, \mathcal{C}] \to [\mathcal{B}, \mathcal{D}]$.

For any pair of objects F, G in $[\mathcal{B}, \mathcal{C}]$, we look at all the morphisms $\alpha : F \Rightarrow G$ and ask if the mapping $\hat{J} : \alpha \mapsto \hat{J}(\alpha)$ is 1–1 and onto. $\hat{J}(\alpha)$ is defined by this rule: for all $B \in \mathcal{B}$, the component $\alpha_B : FB \to GB$ goes to $J(\alpha_B) : JFB \to JGB$. Faithfulness follows immediately: if $\alpha \neq \beta$, then $\alpha_B \neq \beta_B$ for some B, so $J(\alpha_B) \neq J(\beta_B)$ because J is faithful.

Given $\gamma : JF \Rightarrow JG$, is there an $\alpha : F \Rightarrow G$ with $\hat{J}(\alpha) = \gamma$? Because J is full and faithful, for every B there is a unique α_B with $J(\alpha_B) = \gamma_B$. So we use that to define α. Naturality follows because if one of the squares for α didn't commute, we could apply J to it and have a non-commuting square for γ. But γ is a natural transformation, so all its squares commute.

(b) Immediate from Lemma 4.3.8 (p.103, §4.9): since $\hat{J}(G) \cong \hat{J}(G')$ and $\hat{J}$ is full and faithful, it follows that $G \cong G'$.

(c) The adjunctions $F \dashv G$ and $F \dashv G'$ imply that

$$\mathcal{A}(A, GB) \cong \mathcal{B}(FA, B) \cong \mathcal{A}(A, G'B)$$

naturally in A (and in B, but we won't use that until the end). Since $\mathcal{A}(A, GB) = H_{GB}(A)$ and $\mathcal{A}(A, G'B) = H_{G'B}(A)$, we have isomorphisms in **Set**, say $\sigma_A : H_{GB}(A) \cong H_{G'B}(A)$, and these are natural in A. In other words, $H_{GB} \cong H_{G'B}$ in the category of presheaves of $\mathcal{A}$, which is $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, or its subcategory $H_\bullet(\mathcal{A})$. which is shorter to type. This means that $GB \cong G'B$ for all $B \in \mathcal{B}$.

However, $G \cong G'$ says a bit more, namely that G and G' are isomorphic in the functor category $[\mathcal{B}, \mathcal{A}]$. So we apply (b) with $J = H_\bullet$ (where $H_\bullet$ is the Yoneda embedding for $\mathcal{A}$, as above). Just to keep track of the various categories:

$$\widehat{H}_\bullet : [\mathcal{B}, \mathcal{A}] \to [\mathcal{B}, H_\bullet(\mathcal{A})]$$

So we need to check that $\widehat{H}_\bullet(G) \cong \widehat{H}_\bullet(G')$; if so, (b) implies that $G \cong G'$.

Note that $\widehat{H}_\bullet(G) = H_\bullet \circ G$, so

$$\widehat{H}_\bullet(G) : B \mapsto H_{GB}$$

likewise for G', and $H_{GB} \cong H_{G'B}$ for all B, as we've just seen. In other words, for every $B \in \mathcal{B}$ there is an isomorphism α_B in $H_\bullet(\mathcal{A})$ between H_{GB} and $H_{G'B}$. Now we use the naturality in B of the original adjunctions, which carries through to make α_B natural in B. In other words, the α_B's mesh together to form a natural equivalence between $\widehat{H}_\bullet(G)$ and $\widehat{H}_\bullet(G')$. But that means that $\widehat{H}_\bullet(G) \cong \widehat{H}_\bullet(G')$ in the functor category $[\mathcal{B}, H_\bullet(\mathcal{A})]$. So $G \cong G'$.

5 Limits

5.1 5.1.34, p.124

Both diagrams say that $fi = gi$. The equalizer condition says that for any $s : A \to E$ with $fs = gs$, there is a unique $\bar{s} : A \to E$ with $i\bar{s} = s$. The pullback condition says that for any $s_1 : A \to X$ and $s_2 \to Y$ there is a unique $\bar{s} : A \to E$ with $i\bar{s} = s_1$ and $i\bar{s} = s_2$. So we must have $s_1 = s_2$, and the two diagrams impose the same demands. So the answer is yes: equalizer iff pullback.

Moral: if you "pry apart" a diagram by duplicating a node, it doesn't change the diagram's meaning.

5.2 5.1.35, p.124

The pullback lemma fits into the diagram below. Rather than label the arrows, I will indicate compositions by listing nodes along the path; for example, the pullbackness of the left square says that

$$(\forall AC, AE)[ACF = AEF \Rightarrow (\exists! AB)(ABC = AC \;\&\; ABE = AE)]$$

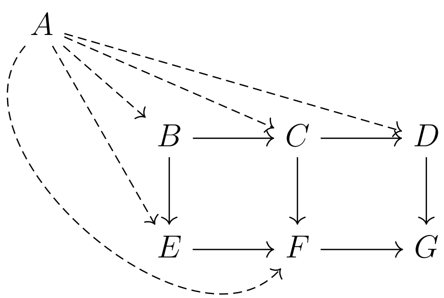
For the right square to be a pullback, we must have

$$(\forall AD, AF)[ADG = AFG \Rightarrow (\exists! AC)(ACD = AD \;\&\; ACF = AF)]$$

and for the whole square,

$$(\forall AD, AE)[ADG = AEFG \Rightarrow (\exists! AB)(ABCD = AD \;\&\; ABE = AE)]$$

We assume that the part of the diagram with solid arrows commutes, but make no assumptions up front about the dashed arrows.



Assume that the left and right square are pullbacks. Pick arbitrary morphisms AD, AE such that $ADG = AEFG$. First use the pullbackness of the right hand square. Let AF be $A(E)F$, the (E) indicating that we're just passing through E, all the right hand square cares about are the nodes AF. We have $ADG = A(E)FG$, again with the right hand square ignoring the (E), so $(\exists AC)[ACD = AD \;\&\; ACF = A(E)F]$. Since $ACF = AEF$, we can use the pullbackness of the left hand square, which tells us that $(\exists AB)[ABC = AC \;\&\; ABE = AE]$. So $ABCD = ACD = AD$. We've shown that the whole square is a pullback, except for the uniqueness requirement. Suppose $\overline{AB}$ is another morphism satisfying $\overline{AB}CD = AD$ and $\overline{AB}E = AE$. We set $\overline{AC} = \overline{AB}C$, and have $\overline{AC}D = \overline{AB}CD = AD$, $\overline{AC}F = \overline{AB}CF = \overline{AB}EF = A(E)F$, i.e., both AC and $\overline{AC}$ satisfy the commutation condition on the right hand square. So by the uniqueness condition for the right hand square, $\overline{AC} = AC$. Now we have $\overline{AB}C = AC$ and $\overline{AB}E = AE$, so by the uniqueness for the left hand square, $\overline{AB} = AB$. Done.

Now assume that the whole square and the right square are pullbacks. Pick AC and AE such that $ACF = AEF$. Then $AEFG = ACFG = ACDG$. So $A(C)DG = AE(F)G$ and by the pullbackness of the whole square, there is an AB such that $ABCD = A(C)D$ and $ABE = AE$. We need to show that $ABC = AC$. By the pullbackness of the right square, there is a unique solution $\overline{AC}$ to the simultaneous equations $\overline{AC}D = AD$ and $\overline{AC}F = AF$, for any AD and AF. Here we let $AD = ACD$ and $AF = AEF$. Then $\overline{AC} = AC$ and $\overline{AC} = ABC$ are both solutions (check!), so $AC = ABC$. It remains to show that AB is unique. Assume $\overline{AB}$ satisfies $\overline{AB}C = AC$ and $\overline{AB}E = AE$. We use the pullbackness of the whole square, which says the solution to $\overline{AB}CD = A(C)D$ and $\overline{AB}E = AE$ is unique. But the "whole square" two equations follow from the "left square" two equations, so we are done.

It's natural to wonder about the "other" pullback lemma: if the left square and the whole square are pullbacks, is the right? The answer is no. Przybylek [10] gives counterexamples, and conditions under which it does hold.

5.3 5.1.36, p.125

(a) We define two cones of shape $\mathbf{I}$ thus: $(A \xrightarrow{h} L \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$ and $(A \xrightarrow{h'} L \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$. Since $p_I h = p_I h'$ for all $I \in \mathbf{I}$, these are the same cone, and the uniqueness requirement on the limit cone says that $h = h'$.

(b) Suppose $h, h' : A \to D_1 \times D_2$, and $h(a)_1 = h'(a)_1$ and $h(a)_2 = h'(a)_2$ for all $a \in A$. Then $h = h'$, i.e., $(h(a)_1, h(a)_2) = (h'(a)_1, h'(a)_2)$ for all $a \in A$.

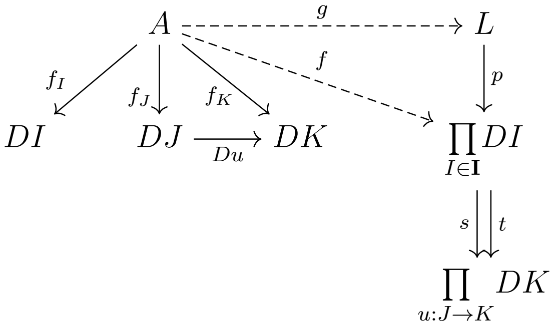
5.4 5.1.37, p.125

Let $(f_I : A \to D(I))_{I\in\mathbf{I}}$ be a cone on D with vertex A. So $(Du)f_I = f_J$ for any $u : I \to J$. Define the function $f : A \to \prod D(I)$ by $a \mapsto (f_I(a))_{I\in\mathbf{I}}$ for all $a \in A$. In fact, f maps A into $\lim\limits_{\leftarrow} D$, because for any $I \xrightarrow{u} J$, $(Du)(f_I(a)) = f_J(a)$ for all $a \in A$, and these are the constraints defining $\lim\limits_{\leftarrow} D$ as a subset of $\prod D(I)$. So range restricting f (and still calling it f), we have the map between vertices. It remains to check that $p_I f = f_I$ for all I, but this is just $p_I f(a) = p_I((f_J(a))_{J\in\mathbf{I}}) = f_I(a)$ for all $a \in A$, which is true. Oh yes, uniqueness of f: $p_I f = f_I$ implies that $f(a)_I = f_I(a)$ for all I, so our choice of f is the only one possible.

5.5 5.1.38, p.125

(a) The proof resembles that given for the special case of **Set** (§5.4), once we bear in mind that the I-component (or u-component) of a morphism is its composition with pr_I (or pr_u). Let $(A \xrightarrow{f_I} DI)_{I\in\mathbf{I}}$ be a cone on D with vertex A. We need to produce a unique $g : A \to L$ with $f_I = p_I g$ for all I.

Here is the key diagram:



The existence and uniqueness of f comes from the definition of the product $\prod DI$. We have:

$$
\begin{aligned}
(\mathrm{pr}_I)f &= f_I && \Pi\ DI \text{ is a product}\\
(Du)f_J &= f_K && (A \xrightarrow{f_I} DI)_{I\in\mathbf{I}} \text{ is a cone}\\
(\mathrm{pr}_u)s &= (Du)\mathrm{pr}_J && \text{definition of } s\\
(\mathrm{pr}_u)t &= \mathrm{pr}_K && \text{definition of } t
\end{aligned}
$$

Now, if $sf = tf$, then by the definition of equalizer, we will have a unique $g : A \to L$ with $pg = f$. Because $\prod_{u:J\to K} DK$ is a product, $sf = tf$ follows from $(\mathrm{pr}_u)sf = (\mathrm{pr}_u)tf$ for all u. We compute:

$$
\begin{aligned}
(\mathrm{pr}_u)sf &= (Du)\mathrm{pr}_J f\\
&= (Du)f_J = f_K\\
(\mathrm{pr}_u)tf &= \mathrm{pr}_K f = f_K
\end{aligned}
$$

So we have a g with $pg = f$. Two computations seal the deal. First, $p_I g = \mathrm{pr}_I pg = \mathrm{pr}_I f = f_I$, as required. Next, suppose $p_I g = p_I g'$, i.e., $\mathrm{pr}_I pg = \mathrm{pr}_I pg'$, for all I. Since $\prod DI$ is a product, $pg = pg'$. Since $p : L \to \prod DI$ is an equalizer, $g = g'$. We have shown there is a unique g with $p_I g = f_I$ for all I, so L is a limit.

(b) This follows from (a) and the fact that binary products plus a terminal object imply all finite products. The empty product is a terminal object (see Example 5.1.9, pp.111–112); the product of one object X is obviously just X; $X \times (Y \times Z)$ satisfies the universal property to be the triple product $X \times Y \times Z$ (easy exercise); the cases $n > 3$ follow by induction.

Smith [11, Ch. 11] gives a lucid and thorough treatment of these results.

5.6 5.1.39, p.125

This exercise follows from §5.5(b) once we show that pullbacks plus a terminal objects implies binary products and equalizers. First recall the remark (p.115) in the definition of pullbacks (Def.5.1.16): when the southeast corner of a pullback square is a terminal object, the pullback is just a binary product. Exercise 5.1.34 (§5.1) shows that all equalizers are pullbacks.

5.7 5.1.40, p.125

First we remark on the fullness of **Monic**(A) as a subcategory of $\mathcal{A}/A$. This simply says that given $X \xrightarrow{m} A$, $X' \xrightarrow{m'} A$ with m and m' monic, we include *all* morphisms $X \to fX'$ of $\mathcal{A}$ making the triangle commute: $m'f = m$. But if $m'f$ is monic, then f has to be monic, by item 6 of §1.11 (a monic composition has a monic on the right).

(a) If $m'f = m$, then $m(X) = m'(f(X)) \subseteq m'(X')$, so if $m \cong m'$ it follows that the images are equal. Conversely, if the images are equal then we can treat m and m' as though they were bijections onto the same set, since monics in **Set** are injective. So we have a bijection between X and X' making the triangle commute.

(b) Subobjects in **Group** are, essentially, subgroups: if $m : H \to G$ is monic then m is a monomorphism and we might as well treat H as being $m(H)$—it's isomorphic to it, and $m(H)$ is a subgroup of G. The same argument works for **Ring** and $\mathbf{Vect}_k$

(c) Subobjects in **Top** of a space X are subsets of X equipped with a topology that is finer than the subspace topology. That is (letting U be the forgetful functor), every subobject of X has a unique representative of the form $i : A \to X$, where Ui is the inclusion map $UA \hookrightarrow UX$, and if $\mathcal{A}$ is the topology of A and $\mathcal{S}$ is the induced subspace topology on UA, then $\mathcal{A} \supseteq \mathcal{S}$. For the proof, see §1.12.

For a *regular* monic, the topology of A will be the subspace topology. See the remark in §5.13.

5.8 5.1.41, p.126

The pullback condition says that

$$(\forall x, y : A \to X)[fx = fy \Rightarrow (\exists! u)(1u = x \& 1u = y)]$$

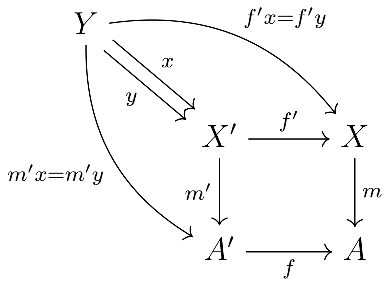
which is clearly equivalent to saying that f is monic.

5.9 5.1.42, p.126

Given $x, y : Y \to X'$ with $m'x = m'y$, we want to show that $x = y$. Since the pullback square commutes, we have

$$mf'x = fm'x = fm'y = mf'y$$

and since m is monic, $f'x = f'y$. So we have this commuting diagram:



Now if we temporarily omit the x and y arrows from the diagram, the pullback condition assures us of an unique morphism $Y \to X'$ making the diagram commute. Since x and y can both play that role, we have $x = y$.

5.10 5.2.21, p.135

Suppose f is an equalizer of s and t, and suppose $g : X \to A$ has $fg = 1_X$, $gf = 1_A$. So we have

$$A \underset{g}{\overset{f}{\rightleftarrows}} X \underset{t}{\overset{s}{\rightrightarrows}} Y$$

Since $sf = tf$, multiply on the right by g and get $sfg = s1_X = s$, $tfg = t1_X = t$, so $s = t$.

On the other hand, suppose that $s = t$ and $f : A \to X$ is an equalizer. Since $s1_X = t1_X$, the definition of equalizer says that 1_X can be factored through f, i.e., there is a $g : X \to A$ with $fg = 1_X$. We must show also that $gf = 1_A$. Now, since $sf = tf$, we have $sfgf = tfgf$, and so fgf can be *uniquely* factored through f, i.e., there is a unique $h : A \to A$ with $fh = fgf$. Obviously $h = gf$ fills the bill. But since $fgf = 1_X f = f$, the equation $fh = fgf$ is the same as $fh = f$ and $h = 1_A$ also fills the bill. By uniqueness, $gf = 1_A$.

The statement about coequalizers is proved by dualizing the argument just given.

5.11 5.2.22, p.135

(a) Define $x \equiv y$ by $(\exists n \in \mathbb{N})[f^n(x) = y \text{ or } f^n(y) = x]$. Another equivalent description: treat X as a graph, where we draw an *undirected* edge from x to $f(x)$ for every $x \in X$. Then the equivalence classes are the connected components of the graph. Finally, the coequalizer is the function taking x to its equivalence class.

(b) The coequalizer in **Top** has the same definition, but now we have to give the quotient set $X/\equiv$ a topology. We use the quotient topology, the finest (largest) topology making the coequalizer continuous. In detail, let $Y = X/\equiv$ and let $e : X \to Y$ be the coequalizer. Then $V \subseteq Y$ is open iff $e^{-1}(V)$ is open in X. Because unions and intersections behave so nicely under inverse functions, this gives a topology.

Let $X = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle. Define f to be a rotation through an angle θ, so $f(e^{it}) = e^{i(t+\theta)}$. It is known that the orbit of f, $\{f^n(z)|n \in \mathbb{Z}\}$, is dense in X for any z if θ is not a rational multiple of 2π (or in degree measure, is an irrational number). So assume that about θ.

Let $e : X \to X/\equiv$ be the coequalizer, and let $Y = (X/\equiv)$. Since each orbit is countable, Y is uncountable. Now let $y \in V \subseteq Y$ with V open in Y. To show that Y is indiscrete, we have to show that $V = Y$. Let $e(x) = y$, so $x \in e^{-1}(V)$. Since V is open, $e^{-1}(V)$ is open in X, so for some $\epsilon > 0$, all points of X within arc distance ϵ of x also belong to $e^{-1}(V)$. But then this is also true for any other x' in $e^{-1}(y)$, for we get x' by rotating x through a multiple of θ, and such a rotation preserves orbits (= equivalence classes): $e(f^n(z)) = e(z)$ for all $z \in X$ and all $n \in \mathbb{Z}$. (Also rotations preserve distances.) Since orbits are dense in X, there is an $x' \in e^{-1}(y)$ within ϵ of any arbitrarily chosen element of X, and so every element of X belongs to

$e^{-1}(V)$. In other words, $e^{-1}(V) = X$ and $V = Y$.

As for the cited fact about density, even more is true: the multiples of an irrational angle are distributed uniformly around the unit circle. Weyl gave a proof using Fourier analysis; there's also a proof using continued fractions. (See Niven [9, §§6.3–6.4] for both proofs.) The density follows from a simple pigeonhole argument. Let the positive integer k be as large as you like, and divide the circle into k equal arcs. So there exists m and n, $m \neq n$, both integers, such that the points at $m\theta$ and $n\theta$ lie in the same arc. So if you rotate the circle through an angle of $-n\theta$, the point at $m\theta$ ends up within (arc distance) $1/k$ of the point at $0°$. But this rotated point isn't *at* $0°$ because θ is irrational in degree measure, and so we cannot have $(m-n)\theta = l \cdot 360°$ for any $l \in \mathbb{Z}$. So there are points in the orbit of θ as close, but not equal, to the point at $0°$ as you like. With another integer multiplication you can put such a point as close to any point on the circle as you like.

§1.12 discusses monics in **Top** in general.

5.12 5.2.23, p.135

(a) Let $i : (\mathbb{N}, +) \hookrightarrow (\mathbb{Z}, +)$ be the inclusion, and suppose $f, g : \mathbb{Z} \to M$ are monoid morphisms such that $fi = gi$. In other words, $f|\mathbb{N} = g|\mathbb{N}$.

Now consider $f(-x)$ for $x \in \mathbb{N}$: $f(-x) + f(x) = f(0) = 0$, and likewise $g(x) + g(-x) = g(0) = 0$. Since $f(x) = g(x)$, writing c for the common value we have

$$f(-x) + c = 0 = c + g(-x)$$

so

$$f(-x) = f(-x) + 0 = f(-x) + c + g(-x) = 0 + g(-x) = g(-x)$$

i.e., $f \equiv g$ over all of $\mathbb{Z}$.

(The last displayed equation is the usual proof that when an inverse exists in a monoid, it's unique. Note that we can't just cavalierly cancel c, since not all monoids have cancellation.)

(b) Let $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion, and suppose $f, g : \mathbb{Q} \to R$ are ring homomorphisms such that $fi = gi$. In other words, $f|\mathbb{Z} = g|\mathbb{Z}$.

Now consider $f(1/y)$ for $y \in \mathbb{Z}$, $y \neq 0$. Since $f(1/y) \cdot f(y) = f(1) = 1$, and likewise $g(y) \cdot g(1/y) = g(1) = 1$, we can write

$$f(1/y) \cdot c = 1 = c \cdot g(1/y)$$

so

$$f(1/y) = f(1/y) \cdot 1 = f(1/y) \cdot c \cdot g(1/y) = 1 \cdot g(1/y) = g(1/y)$$

and so

$$f(x/y) = f(x) \cdot f(1/y) = g(x) \cdot g(1/y) = g(x/y)$$

for any $x, y \in \mathbb{N}$ with $y \neq 0$. So $f = g$ over all of $\mathbb{Q}$.

5.13 5.2.24, p.135

This exercise is the dual of §5.7, sort of. We start by remarking on the fullness of $\mathbf{Epic}(A)$ as a subcategory of $A/\mathcal{A}$. This simply says that given $A \xrightarrow{e} X$, $A \xrightarrow{e'} X'$ with e and e' epic, we include *all* morphisms $X \to fX'$ of $\mathcal{A}$ making the triangle commute: $fe = e'$. But if fe is epic, then f has to be epic, by item 7 of §1.11 (an epic composition has an epic on the left).

(a) The equivalence relations induced by e and e' are $e(x_1) = e(x_2)$ and $e'(x_1) = e'(x_2)$. Suppose we have a morphism $f : X \to X'$ with $fe = e'$; then $e(x_1) = e(x_2) \Rightarrow fe(x_1) = fe(e_2) \Rightarrow e'(x_1) = e'(x_2)$. On the other hand, suppose $e(x_1) = e(x_2) \Rightarrow e'(x_1) = e'(x_2)$; then we define $f : X \to X'$ by $f(x) = e'(e^{-1}(x))$, which is well-defined because the e-inverse image of x is sent to a single element of X' by e, because e is surjective and $e(x_1) = e(x_2)$ $\Rightarrow e'(x_1) = e'(x_2)$. It is clear that with this definition, $fe = e'$.

It follows that if $e(x_1) = e(x_2) \Leftrightarrow e'(x_1) = e'(x_2)$, then we have functions $f : X \to X'$ and $f' : X' \to X$ with $f'f$ and ff' as solutions to $ue = e = 1_X e$ and $u'e' = e' = 1_{X'}e'$. Since e and e' are epic, $f'f$ and ff' are both identities, so e and e' are isomorphic. Conversely, if e and e' are isomorphic, then we have $e(x_1) = e(x_2) \Leftrightarrow e'(x_1) = e'(x_2)$, so the equivalence relations are the same.

To cap the argument off, we need to know that for every equivalence relation on A, there is an epic $e : A \to X$ such that $e(x_1) = e(x_2)$ is that equivalence relation. We let X be the set of equivalence classes and let $e(x)$ be the equivalence class of x.

(b) The argument given in (a) works without modification up to the final paragraph (assuming that all epics in **Group** are surjective; see §1.13 for a proof). It shows that (group) epimorphisms $e : A \to X'$ and $e' : A \to X'$ are isomorphic iff they define the same equivalence relation. However, not every equivalence relation on A has an associated epimorphism. A equivalence relation $\equiv$ is called a **congruence** when it satisfies the additional requirement that $x_1 \equiv x_2$ and $y_1 \equiv y_2$ imply $x_1y_1 \equiv x_2y_2$. It's trivial to show that if e is an epimorphism, then $e(x_1) = e(x_2)$ defines a congruence; conversely, given a congruence on A, the set of congruence classes forms a group under the obvious definition of a group operation, with the function $e(x) =$ the congruence class of x being an epimorphism.

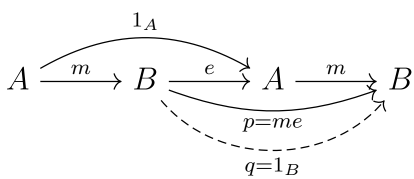
We now observe that for an epimorphism e, $e(x) = e(y)$ is equivalent to $e(xy^{-1}) = 1$, which is (by definition) the same as saying that xy^{-1} is in the kernel of e. We clinch the argument with the claim that a subgroup of A is a kernel if and only if it's a normal subgroup. This is standard basic group theory (see Lang [5, §1.3] or Bergman [2, §§4.1–4.2]), but is also an easy exercise.

Leinster remarks, "Arguably, quotient object would be more suitable for an isomorphism class of *regular* epics…" §1.12, on monics and epics in **Top**, suggests his meaning. A monic in **Top** is a continuous injective function, so the subspace topology on the image can be (strictly) smaller than the image topology (i.e., the one that makes the image homeomorphic to the source space). An epic in **Top** is a continuous surjective function, so the quotient topology on the image can be (strictly) larger than the topology of the target space. With regular monics and epics, this can't happen. If 'subobject' should mean "up to isomorphism, a subspace", and 'quotient' should mean quotient space in the sense found in most topology textbooks (see Munkres [8, §22], or §1.12), then regular monics and epics serve these up.

5.14 5.2.25, p.135

See §1.11 for background on monics.

(a) First suppose m is split monic with $em = 1_A$. We let $C = B$, $p = me$, and $q = 1_B$, thus:

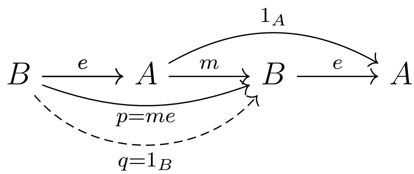


(Warning: the diagram doesn't commute, but it would if you removed the 1_B arc; that's why I've made it dashed.) We have $pm = mem = m1_A = m = 1_Bm = qm$. Now assume $f : X \to B$ satisfies $pf = qf$, so $f = 1_Bf = mef$. So f factors through m via ef, i.e., setting $u = ef$ gives $f = mu$. We have to show that u is unique. From $f = mu$ we get $ef = emu = 1_Au = u$, so $u = ef$ is the only possible solution. Thus m is regular monic.

Next, suppose m is regular monic with $m : A \to B$ equalizing $p, q : B \to C$.

Suppose $x, y : X \to A$ with $mx = my = f$, say. Since $pm = qm$, we have $pmx = qmx$, i.e., $pf = qf$; likewise $pmy = qmy$, also giving $pf = qf$. Since m is an equalizer, f must factor through m uniquely, i.e., there is a unique $u : X \to A$ satisfying $mu = f$. But $x = u$ and $y = u$ both satisfy this equation, so we must have $x = y$. Therefore m is monic.

Although Leinster doesn't ask for it, here are the dual arguments, showing that split epic ⇒ regular epic ⇒ epic. First split epic ⇒ regular epic.



(Same warning: the diagram doesn't commute, but would if the 1_B arc were removed, which is why it's dashed.) We have $ep = eme = 1_A e = e = e1_B = eq$. If $f : B \to X$ satisfies $fp = fq$, i.e., $f = f1_B = fme$, then setting $u = fm$ we have $f = ue$. To show e is regular epic we need to show finally that u is unique. But if $f = ue$, then $fm = uem = u1_A = u$.

Next regular epic ⇒ epic. Suppose e is regular epic with $e : B \to C$ coequalizing $p, q : A \to B$. Suppose $x, y : C \to X$ with $xe = ye = f$, say. Since $ep = eq$, we have $xep = xeq$ giving $fp = fq$; we could have used y instead of x, giving $yep = yeq$ which also says that $fp = fq$. Since e is a coequalizer, f must factor through e uniquely, i.e., there is a unique $u : C \to X$ satisfying $ue = f$. Since both $x = u$ and $y = u$ satisfy this equation, we must have $x = y$. Therefore e is epic.

(b) Suppose $f : G \to H$ in **Abelian**. We prove the implications f monic ⇒ ($\ker f = 0$) ⇒ f regular monic. (These are really equivalences because regular monic implies monic, as we just saw in (a).)

f monic ⇒ ($\ker f = 0$): set $K = \ker f$, so we have two maps: the zero map $0 : K \to G$ sending all of K to $0 \in G$, and the inclusion map $i : K \hookrightarrow G$. Now $fi = f0$, both being the zero map from K to H. If f is monic then we can cancel it from $fi = f0$, so $i = 0$, i.e., $\ker f = 0$.

($\ker f = 0$) $\Rightarrow$ f regular monic: let $s : H \to H/f(G)$ be the canonical epimorphism. Consider

$$G \xrightarrow{f} H \underset{0}{\overset{s}{\rightrightarrows}} H/f(G)$$

where 0 is the zero map. We immediately have $sf = 0f$, and $s(x) \neq 0$ if $x \notin f(G)$. So if $g : L \to H$ has $sg = 0g$, then $g(L) \subseteq \ker s = f(G)$. If $\ker f = 0$ then f is injective, so f determines an isomorphism between G and $f(G)$, and so g can be "pulled back along f" to give a map $r : L \to G$. That is, we have a unique $r : L \to G$ with $fr = g$. Thus f is an equalizer.

Finally, an example of a regular monic that is not split. The standard example is the inclusion $i : 2\mathbb{Z} \hookrightarrow \mathbb{Z}$. This has kernel 0, so is regular monic. Suppose $e : \mathbb{Z} \to 2\mathbb{Z}$. For any $k \in \mathbb{Z}$, $e(k) = k \cdot e(1)$—for $k > 0$, write k as a sum of k 1's, and for $k < 0$, we have $e(-k) = -e(k)$ from basic group theory. So $e(2) = 2 \cdot e(1)$. Now, if ei is the identity on $2\mathbb{Z}$, then we must have $e(2) = ei(2) = 2$. But that is inconsistent with the equation $e(2) = 2 \cdot e(1)$: if $2 = e(2) = 2 \cdot e(1)$, then we cannot have $e(1) \in 2\mathbb{Z}$.

(c) As shown in §1.12, the monics in **Top** are, up to isomorphism, subsets equipped with spaces that are finer than (larger or equal to) the induced subspace topology. The regular monics are, up to isomorphism, just the subspaces. So any map $i : A \to X$ where $UA \subseteq UX$ and the topology of A is (strictly) larger than the subspace topology serves as an example of a non-regular monic.

5.15 5.2.26, p.136

See §1.11 for background on monics and epics.

(a) By definition, isomorphism $\Rightarrow$ split epic plus split monic; item 12 of

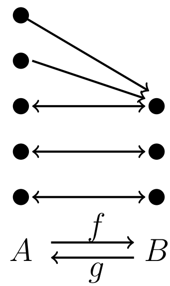


Figure 3: Monic and Epic in **Set**

§1.11 says that split epic ⇒ regular epic, and item 11 that split monic ⇒ monic, so we have the forward implication.

Item 3 shows that split epic plus monic ⇒ isomorphism, so it's enough to prove that regular epic plus monic ⇒ split epic. Suppose $f : A \to B$ coequalizes $p, q : X \to A$. Then $fp = fq$; since f is monic, $p = q$. The other part of being a coequalizer says that if $h : A \to Y$ satisfies $hp = hq$, then there exists a $g : B \to Y$ with $gf = h$. Apply this with $h = 1_A : A \to A$; $1_A p = 1_A q$ because as we just saw, $p = q$. So $gf = 1_A$ and f is split epic.

Dualizing all this shows that isomorphism ⇔ split monic plus epic.

(b) In **Set**, epic and split epic are both equivalent to surjective. This is basic set theory; we start with the implication surjective ⇒ split epic. Rather than write out a formal proof, I invite you to look at figure 3.

If $f : A \to B$ is surjective, then we have all the → arrows, and for each $b \in B$, we choose one incoming arrow to reverse. This gives us the ← arrows. (We appeal here to the axiom of choice.) Clearly we have $fg = 1_B$, so f is split epic.

Next, epic ⇒ surjective, which we prove in contrapositive form. If $f : A \to B$ is not surjective, then we let $p : B \to \{0, 1\}$ be the characteristic function of $f(A)$, and $q : B \to \{0,1\}$ be the constant function $q(b) \equiv 1$. Then $pf = qf$ (constantly 1), but $p \neq q$ (because f is not surjective). So f is not epic.

Since surjective $\Rightarrow$ split epic $\Rightarrow$ regular epic $\Rightarrow$ epic $\Rightarrow$ surjective, these are all equivalent. (The middle implications come from dualizing §5.14(a); details below. Note, however, that the implication split epic $\Rightarrow$ epic is nearly trivial.)

This is a good spot to prove the analogous facts for monics in **Set**. We have split monic $\Rightarrow$ regular monic $\Rightarrow$ monic from §5.14(a), so our next step is monic $\Rightarrow$ injective. We prove the contrapositive. Suppose $g : B \to A$ is not injective, with $g(b) = g(b')$. Let $p, q : \{0\} \to B$ be defined by $p(0) = b$, $q(0) = b'$. Then $gp = gq$ but $p \neq q$.

Finally, we'd like to show injective $\Rightarrow$ split monic, but there is a class of exceptions: $\varnothing \to A \neq \varnothing$. If $B \neq \varnothing$ and $g : B \to A$ is injective, then referring to our diagram above, we have the $\leftarrow$ arrows. Reversing them gives us a partial function $f : A \to B$ with $fg = 1_B$. To make f a true function, we pick a $b_0 \in B$ and send all "orphans" (i.e., $a \in A \smallsetminus g(B)$) to b_0. So g injective with nonempty domain $\Rightarrow$ split monic. (Picking a single element from a nonempty set doesn't require the axiom of choice, but follows from the basic rules of logic.)

It remains to prove that injective $\Rightarrow$ regular monic. We just need to deal with case $\varnothing \to A \neq \varnothing$, showing that such functions are equalizers. Let $p, q : A \to \{0,1\}$ be the constant functions $p(x) \equiv 1$, $q(x) \equiv 0$. Then $\varnothing \to A$ is the equalizer of p and q.

(c) In **Group**, the simplest example of a non-split epic is $f : \mathbb{Z} \to \mathbb{Z}_n$. (Note: $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.) The only homomorphism $\mathbb{Z}_n \to \mathbb{Z}$ is the trivial one sending everything to 0, so the equation $fi = 1_{\mathbb{Z}_n}$ is impossible.

For **Top**, two examples of non-split regular epics are given in §1.12. First, the covering map of the circle $\mathbb{R} \to S^1$, given by $x \mapsto e^{ix}$. Second, the same

	Monic	Regular Monic	Split Monic	Epic	Regular Epic	Split Epic
Composition	✓	×	✓	✓	×	✓
Pullbacks	✓	×	×	×	×	✓

Figure 4: Compositions and Pullbacks

map but restricted to the closed interval $[0, 2\pi]$.

Worth noting, perhaps: the covering map $\mathbb{R} \to S^1$ becomes the group theory example upon restricting the domain to $2\pi\mathbb{Z}$.

5.16 5.2.27, p.136

The results are summarized in fig.4. First we look at composition, then pullbacks.

Items (4) and (5) of §1.11 tell us that monics, epics, split monics, and split epics are all closed under composition.

A composition of regular monics need not be regular. It *is* regular in **Top**, **Hausdorff**, **CptHff**, and **Group**, as can be seen from the results in §1.12 and §1.13. The following example comes from Adámek et al. [1, §7J]. We say a topological space X is **functionally Hausdorff** if every pair of distinct point can be separated by a real-valued function, i.e., for any $a, b \in X$, $a \neq b$, there is a continuous $f : X \to \mathbb{R}$ such that $f(a) \neq f(b)$. **FHaus** is the full subcategory of **Hausdorff** consisting of all functionally Hausdorff spaces and all continuous functions between them.

First we characterize regular monics in **FHaus**. The argument in §1.12 shows that f is a regular monic iff it is a homeomorphism onto a *solution set*, i.e., a set of the form $S = \{x \in X | s(x) = t(x)\}$ for some $s, t : X \to Y$.

A **zeroset** is $f^{-1}(0)$ for some continuous real-valued f; we show next that solution sets are the same as intersections of zerosets. First suppose S is a solution set for s, t, as above. For each $x \notin S$, we have $s(x) \neq t(x)$ and so there is an $f : Y \to \mathbb{R}$ with $f(s(x)) \neq f(t(x))$, and so $g(x) \stackrel{\text{def}}{=\!=} f(s(x)) - f(t(x))$ is a real-valued map with $g(x) \neq 0$. On the other hand, $g(S) = 0$ (i.e., $(\forall x \in S)g(x) = 0$). So if we consider the family of all $g : X \to \mathbb{R}$ for which $g(S) = 0$, then S is the intersection of the family of associated zerosets.

In the other direction, let S be an intersection of zerosets, say $S = \bigcap_\alpha \{x \in X | f_\alpha(x) = 0\}$ for some family $\{f_\alpha : X \to \mathbb{R}\}$. Let $Y = \prod_\alpha \mathbb{R}$, the product of as many copies of $\mathbb{R}$ as there are maps in the family. Define $s : X \to Y$ by $s(x)_\alpha = f_\alpha(x)$, i.e., the α component of $s(x)$ is given by $f_\alpha(x)$. Let t be the constantly 0 map, i.e., $t(x)_\alpha = 0$ for all α. It is clear that S is the solution set for s and t.

Now we consider $X \subseteq Y \subseteq R$ in **FHaus**, where X is $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with the discrete topology, Y is $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} = X \cup \{0\}$ with the discrete topology, and R is $\mathbb{R}$ except that we add one new closed set to the standard topology of $\mathbb{R}$, namely X. Once you add X, you have to add all finite unions of closed sets, in other words all sets of the form $F \cup X$ for F closed in the standard topology. The family of sets $\{F | F \text{ closed in } \mathbb{R}\} \cup \{F \cup X | F \text{ closed in } \mathbb{R}\}$ is closed under arbitrary intersections, so it's the topology of R specified via closed sets. The open sets in R are thus $\{V | V \text{ open in } \mathbb{R}\} \cup \{V \setminus X | V \text{ open in } \mathbb{R}\}$. Discrete spaces such as X and Y obviously belong to **FHaus**; likewise, since $\mathbb{R}$ is in **FHaus** and R has a finer topology than $\mathbb{R}$, R is also in **FHaus**.

We have the set-theoretic inclusion maps $X \to Y \to R$ (i.e., $UX \hookrightarrow UY \hookrightarrow UR$); we note next that these are inclusion maps in **FHaus**, i.e., imbeddings. The maps $X \to Y \to R$ are continuous because X and Y are discrete, and $X \to Y$ is obviously a homeomorphism onto its image. Now, $Y \subseteq \mathbb{R}$ with the subspace topology is already "close to" being discrete: the only singleton that's not an open set is $\{0\}$. We made X closed in R, so $Y \setminus X = \{0\}$ is

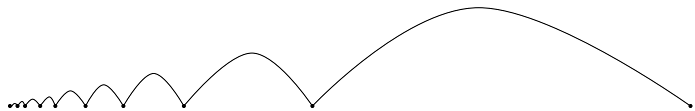


Figure 5: **FHaus** example

open in $Y \subseteq R$ with the subspace topology. So $X \hookrightarrow Y \hookrightarrow R$.

We define X as a zeroset in Y via $f(X) = 0$, $f(0) = 1$. So $X \hookrightarrow Y$ is regular monic. It's easy enough to define Y as a zeroset in R, or even in $\mathbb{R}$: for example, define $g : R \to \mathbb{R}$ by $g(r) =$ the distance from r to the closest element of Y (a piecewise linear function, except at 0). So $Y \hookrightarrow R$ is also regular monic.

To finish the example, we demonstrate that $X \subseteq R$ is not an intersection of zerosets, and so $X \hookrightarrow R$ is not regular monic. This follows from the fact that if $h : R \to \mathbb{R}$ is continuous, and $h(X) = 0$, then $h(0) = 0$. (Loosely speaking, even though 0 is not in the closure of X in R, it still acts like it is, so far as real-valued functions are concerned. See fig.5.) Proof: suppose on the contrary that $h(0) = r \neq 0$ but $h(X) = 0$. Adapting the good old ϵ-δ definition of continuity to a map from R to $\mathbb{R}$:

$$(\forall \epsilon > 0)(\exists U \ni 0)(\forall x)[x \in U \Rightarrow |h(x) - r| < \epsilon] \qquad U \text{ open in } R$$

We choose $\epsilon < |r|/2$. We described the open sets in R above; say $U = V \setminus X$ for V open in $\mathbb{R}$. This tells us that $h(x)$ is bounded away from 0 for all x "near" 0 but not in X, or more precisely:

$$(\exists \delta > 0)(\forall x)[|x| < \delta \,\&\, x \notin X \Rightarrow |h(x)| > |r|/2]$$

But $h(x_0) = 0$ for all $x_0 \in X$, and there is certainly an $x_0 \in X$ with $|x_0| < \delta$. Then h cannot be continuous at that x_0, since for all x sufficiently near but not equal to x_0, $|h(x)| > |r|/2$.

As fig.5 suggests, the fact that $h(1/n) = 0$ for all $n > 0$ forces $|h(x)|$ to be small for all x in between $\frac{1}{n}$ and $\frac{1}{n+1}$; the larger n, the smaller $|h(x)|$ for

$\frac{1}{n+1} < x < \frac{1}{n}$. So $\lim |h(x)| = 0$ as $x \to 0$, even with x avoiding the points $1/n$.

A composition of regular epics need not be regular. It *is* regular in **Top**, **Hausdorff**, **CptHff**, and **Group**, as can be seen from the results in §1.12 and §1.13. Perhaps surprisingly, regular epics are also closed under composition in **FHaus**. The arguments in §1.12 for **Hausdorff** adapt virtually without change[5] to show that the regular epics in **FHaus** are precisely the quotient maps. Quotient maps are closed under composition; this is a one-line proof.

FHaus$^{\text{op}}$ comes to the rescue, exhibiting regular epics whose composition is not regular. Proof: regular epics in **FHaus**$^{\text{op}}$ are regular monics in **FHaus** and vice versa. Because this may seem like cheating, here is an example from Adámek et al. [1, §7S, p.131], in **Cat**. (I don't know of a topological example.) We have three categories $\mathcal{A} \xrightarrow{A} \mathcal{B} \xrightarrow{B} \mathcal{C}$ with morphisms (i.e., functors) A and B; A and B are regular epic but BA is not.

The categories:

1. $\mathcal{A}$ is $0 \xrightarrow{a} 1$, i.e., two objects with one morphism between them. (Also the identity morphisms, of course.)

2. $\mathcal{B}$ is $(\mathbb{N}, +)$ as a category. So there is one object, call it $*$, and morphisms $\{b_n | n \in \mathbb{N}\}$ with $b_i b_j = b_{i+j}$. Note that $b_0 = 1_*$.

3. $\mathcal{C}$ is $(\{0, 1\}, \cdot)$ as a category. So there is one object, call it $\circ$, and morphisms c_0, c_1 with $c_i c_j = c_{i \cdot j}$. Note that $c_1 = 1_\circ$.

[5]The "abstract" proof that $ij = 1_Y$ still works, although the "dense image" one no longer does.

The functors A and B are defined by:

$$\begin{aligned} A1_0 &= A1_1 = b_0 = 1_* \\ Aa &= b_1 \\ Bb_0 &= c_1 = 1_\circ \\ Bb_n &= c_0 \text{ for } n > 0 \end{aligned}$$

Proof that A is regular epic: let $\mathbf{1}$ be the category with one object (call it †) and just the identity morphism. Then A is the coequalizer of $P, Q : \mathbf{1} \to \mathcal{A}$ where $P\dagger = 0$ and $Q\dagger = 1$. Checking this is routine, but amounts to noting that if $K : \mathcal{A} \to \mathcal{D}$ satisfies $KP = KQ$, then $K0 = K1$, and Ka is some morphism from $K0$ to $K1$. The functor A imposes no "conditions" on Aa—nothing like, say, $(Aa)(Aa) = 1_*$—so if we define a functor $\mathcal{B} \to \mathcal{D}$ by $b_n \mapsto (Ka)^n$, everything is copacetic[6].

Proof that B is regular epic: let $A' : \mathcal{A} \to \mathcal{B}$ be defined by $A'a = b_2$. Trivially we have $BA = BA'$. Suppose $L : \mathcal{B} \to \mathcal{E}$ satisfies $LA = LA'$. So $Lb_1 = Lb_2 = \ell$, say. Now $b_1 b_1 = b_2$, so $\ell\ell = (Lb_1)(Lb_1) = Lb_2 = \ell$, so by induction $\ell^n = \ell$ for all $n > 0$. This observation makes it easy to see that B coequalizes A and A'.

Proof that BA is not regular: suppose BA coequalizes $S, T : \mathcal{Z} \to \mathcal{A}$. So $BAS = BAT$, and if $F : \mathcal{A} \to \mathcal{F}$ satisfies $FS = FT$, then we have a unique solution X to $XBA = F$. Suppose first that for every object $z \in \mathcal{Z}$, we have $Sz = Tz$. This forces $S = T$, because if f is the morphism $z_1 \xrightarrow{f} z_2$ in $\mathcal{Z}$, then there is only one possible choice for Sf and Tf, namely either a or one of the identities 1_0 or 1_1. So if $F : \mathcal{A} \to \mathcal{A}$ is the identity functor, then $FS = FT$. But the equation $XBA = F$ is impossible because BA sends 0 and 1 to the same object $\circ$, while F keeps them distinct. (This also works for $\mathcal{Z}$ the empty category, where $S = T =$ the empty functor.)

Next, suppose that $Sz \neq Tz$ for some $z \in \mathcal{Z}$. We let $F = A$ (and so $\mathcal{F} = \mathcal{B}$). For all objects $z \in \mathcal{Z}$, it's obvious that $ASz = ATz$. For a

[6] fine, hunky-dory, A-OK, etc.

morphism $z_1 \xrightarrow{f} z_2$, there are two possible cases for $BASf$ (and it's the same story for $BATf$):

$$\begin{cases} Sz_1 = 0 \,\&\, Sz_2 = 1 \Rightarrow Sf = a \Rightarrow ASf = b_1 \Rightarrow BASf = c_0 \\ Sz_1 = Sz_2 \Rightarrow Sf = (1_0 \text{ or } 1_1) \Rightarrow ASf = b_0 \Rightarrow BASf = c_1 \end{cases}$$

So if $BASf = BATf$, then $ASf = ATf$. But $BAS = BAT$ by hypothesis, so $AS = AT$. Thus we must have X with $XBA = A$. Applying this to the morphism a in $\mathcal{A}$, $b_1 = Xc_0$. However, this leads to the contradiction $b_2 = b_1 b_1 = (Xc_0)(Xc_0) = X(c_0 c_0) = Xc_0 = b_1$, not true in $\mathcal{B}$. Therefore BA does not coequalize S and T.

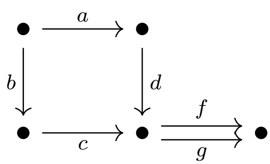
Side comment: Adámek et al. [1, §7.40(6), p.112] remark that the functor A is a non-surjective epic in **Cat**. Now there are two forgetful functors **Cat** $\to$ **Set**, namely Ob, the set of objects, and Mor, the set of morphisms. Ob(A) is surjective, as it has to be: Ob has a right adjoint (§3.4, Exercise 3.2.16, p.78), and functors with right adjoints preserve epics (item 10 of §1.11). But Mor(A) is clearly not surjective.

Next, pullbacks. §5.8 (Exercise 5.1.41) proves that monics are pullback stable. Split monics are not: Example 5.1.17(b) (p.117) demonstrates that the pullback of inclusions $X, Y \hookrightarrow Z$ in **Set** is the intersection $X \cap Y \hookrightarrow X, Y$. If X and Y are nonempty but disjoint, then the pullback of $Y \hookrightarrow Z$ along $X \hookrightarrow Z$ is not split, being $\varnothing \hookrightarrow X$, and the only non-split monics in **Set** are empty functions to a nonempty set.

Example 5.1.17(b) also holds true for **Hausdorff**, as is readily checked. We know from §1.12 that epics in **Hausdorff** are maps with dense images. So this diagram shows an epic that is not pullback stable:

$$\begin{array}{ccc} \varnothing & \hookrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \mathbb{R} \smallsetminus \mathbb{Q} & \hookrightarrow & \mathbb{R} \end{array}$$

For a regular monic whose pullback is not regular, we resort to a technique useful for many counterexamples: a custom-built category. First we express the essence of the problem in a diagram:



To make this into a category, we must have, at a minimum, five identities plus the following additional morphisms:

$$da, cb, fc, gc, fd, gd, fcb, gcb, fda, gda$$

In other words, a morphism for every possible path. However, this set of 21 morphisms suffice. They don't have to all be distinct: we can declare, for example, that $fc = gc$, even though $f \neq g$. Certain equalities imply other equalities; e.g., $fc = gc$ implies $fcb = gcb$. But we have great freedom in deciding which equalities should hold.

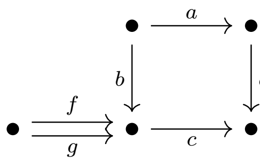
We declare the following equalities:

$$da = cb, \quad fc = gc, \quad fcb = gcb$$

but otherwise none should hold except for the trivial ones involving the identities. It is readily checked that in this category, c is an equalizer, a is a pullback of c, and a is not an equalizer[7]. So the pullback of the regular monic c is not regular.

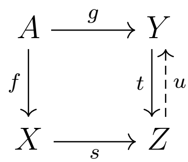
[7] Caveats: we have to consider possible solutions of $xa = ya$ for *all* pairs x, y. It's not enough to note that only one arrow leads out of the target of a; $x = fd$ and $y = gd$ also count. So do pairs like $x = y = d$. Now, if D is the source of d, then 1_D is *an* equalizer of the $d = d$ pair. But if a were a unique isomorphism, it would also be an equalizer of $d = d$. Since a isn't an isomorphism, we're good.

In a similar fashion, the diagram

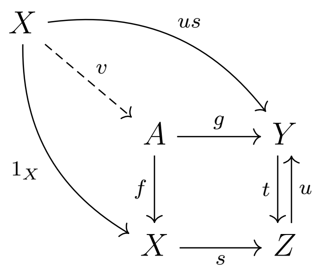


provides the foundation of a category in which c is a coequalizer but its pullback a isn't. We declare $da = cb$ and $cf = cg$, but no other nontrivial equalities hold.

Surprisingly, split epics are pullback stable. Suppose t in this pullback square is split epic, with $tu = 1_Z$:



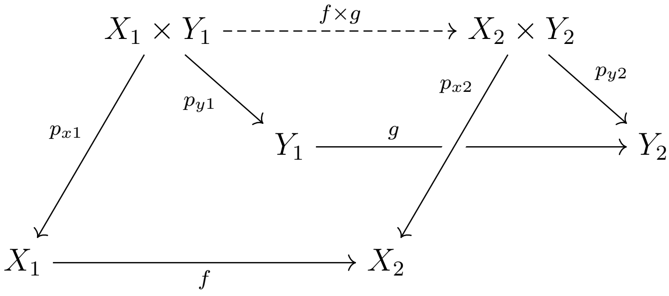
(Possibly $ut \neq 1_Y$, but otherwise the diagram commutes.) We need to find a right inverse for f. Now $tus = s = s1_X$, so since our square is a pullback, there must exist a unique v making the left and top sides of this diagram commute:



So $fv = 1_X$ and we are done.

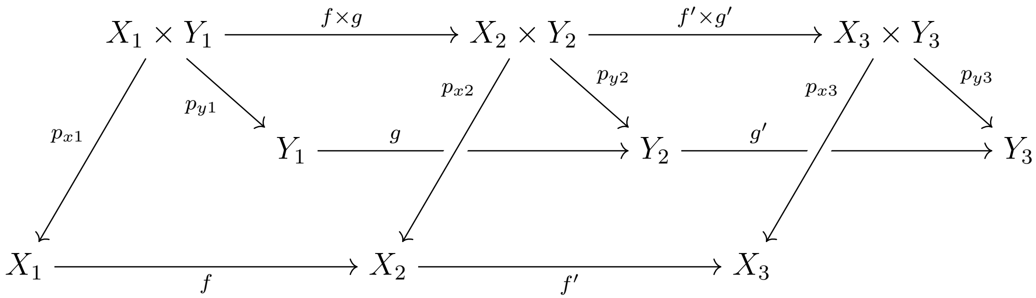
5.17 5.3.8, p.140

Suppose that binary products $X_1 \times Y_1$ and $X_2 \times Y_2$ have been chosen for the pairs (X_1, Y_1) and (X_2, Y_2), and suppose we have a morphism $(f, g) : (X_1, Y_1) \to (X_2, Y_2)$ in the category $\mathcal{A} \times \mathcal{A}$. We need to define a morphism $f \times g$ from $X_1 \times Y_1$ to $X_2 \times Y_2$. Consider this diagram:



(I'm using p_{x1}, etc., instead of Leinster's $p_1^{X_1,Y_1}$, etc.) We have morphisms from $X_1 \times Y_1$ to X_2 and Y_2 obtained by going "down and across", namely fp_{x1} and gp_{y1}. Since $X_2 \times Y_2$ is a product, there is a unique morphism $f \times g$ making the diagram commute.

To show that $(f, g) \mapsto f \times g$ is functorial, chase this diagram:

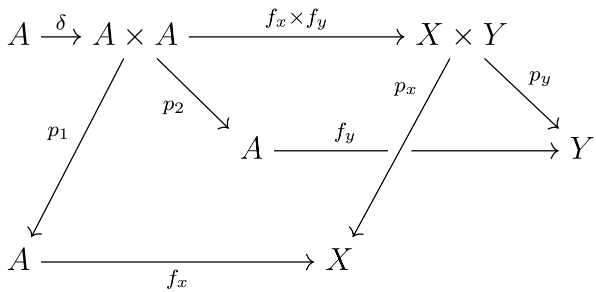


Hence $(f' \times g')(f \times g) = (f'f) \times (g'g)$.

5.18 5.3.9, p.140

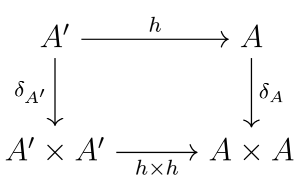
Given a morphism $f : A \to X \times Y$, define the pair (f_x, f_y) in $\mathcal{A}(A, X) \times \mathcal{A}(A, Y)$ by the compositions $f_x = p_x f$, $f_y = p_y f$, where p_x and p_y are the projections. In the reverse direction, given $(f_x, f_y) \in \mathcal{A}(A, X) \times \mathcal{A}(A, Y)$, by the definition of product there is a unique $f : A \to X \times Y$ with $f_x = p_x f$ and $f_y = p_y f$. So we have the posited 1–1 correspondence.

A good way to think of this correspondence is as a special case of the first diagram in §5.17, with the diagonal morphism prepended. (This is the unique morphism $\delta : A \to A \times A$ such that $p_1\delta = p_2\delta = 1_A$, with p_1 and p_2 being the two projections from $A \times A$ to A.) Thus:

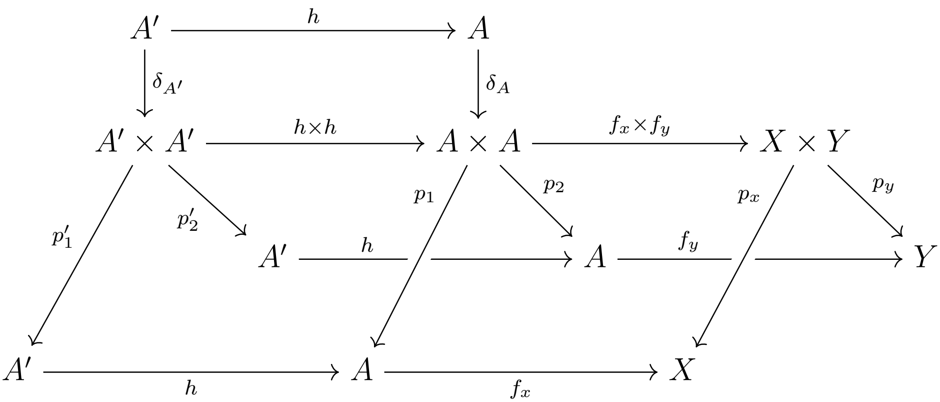


Since $f_x p_1 \delta = f_x$ and $f_y p_2 \delta = f_y$, the map $(f_x, f_y) \mapsto f$ is just $(f_x, f_y) \mapsto (f_x \times f_y)\delta$. The naturality in X and Y then falls out from the functoriality of $(f_x, f_y) \mapsto f_x \times f_y$.

The naturality in A follows from the naturality of the diagonal map $A \mapsto \delta_A$, namely the commutativity of this diagram for any $h : A' \to A$:



We check this by composing with the projections $p_1, p_2 : A \times A \to A$. We have $p_1 \delta_A h = 1_A h = h$ and $p_1(h \times h)\delta_{A'} = h p'_1 \delta_{A'} = h 1_{A'} = h$, likewise for composing with p_2, so the diagram commutes. We then have this version of the second diagram from §5.17:



What's the point of the diagram? Well, if we start with the pair (f_x, f_y), this is mapped to $(f_x \times f_y)\delta_A$. Likewise, $(f_x h, f_y h) \mapsto (f_x h \times f_y h)\delta_{A'}$. But from the diagram, we have this relation between the two values:

$$(f_x h \times f_y h)\delta_{A'} = (f_x \times f_y)(h \times h)\delta_{A'} = (f_x \times f_y)\delta_A h$$

and this is exactly what naturality in A means.

5.19 5.3.10, p.140

Suppose $F : \mathcal{A} \to \mathcal{B}$ creates limits of shape $\mathbf{I}$ and suppose $D : \mathbf{I} \to \mathcal{A}$ is a diagram.

Using Def.5.3.5 (p.139) of "create limits", the first bullet says that for any limit cone M in $\mathcal{B}$ on FD, there is a *unique* cone L on D in $\mathcal{A}$ with $FL = M$—limit cones in $\mathcal{B}$ "lift uniquely" to cones in $\mathcal{A}$. The second bullet says that the lifted cone is a limit cone. So if we have a cone L on D producing a limit cone FL on FD, then the unique lifted cone is L, which must be a limit cone.

The "more healthy and inclusive" notion of "creates limits", given by Leinster at the top of p.140, has two clauses; the second says that for *every* cone L on D, if FL is limit cone then L a limit cone ("every such cone is itself a limit cone"). This is precisely the meaning of "reflects limits" for cones on D. We can rephrase the "healthy" notion this way: F creates limits of shape $\mathbf{I}$ iff F reflects limits of shape I, and if there exists a limit cone on FD then there is one that lifts to a cone on D (which then must be a limit cone).

Some useful and common terminology for a functor $F : \mathcal{A} \to \mathcal{B}$: if D is a diagram in $\mathcal{A}$ and M is a cone on FD, then M **lifts** to a cone L on D iff $FL = M$. F **lifts limits** of shape $\mathbf{I}$ iff every limit cone on a diagram FD of shape $\mathbf{I}$ can be lifted to a limit cone. F **lifts limits uniquely** of shape $\mathbf{I}$ if in addition these lifts are unique, i.e., if L and L' are limit cones lifting M, then $L = L'$. F **lifts limits** iff it lifts limits of all shapes; likewise for **lifts limits uniquely**.

(We don't talk about lifting *arbitrary* cones in $\mathcal{B}$: if any one of the objects or morphisms of the cone wasn't the image of an object or morphism of $\mathcal{A}$, we'd be dead in the water. Also note that the uniqueness in *lifts limits uniquely* derives from two pieces of information: first that $FL = FL'$, second that L and L' are both limit cones. In Def.5.3.5, the uniqueness flows just from $FL = FL'$, plus FL being a limit cone.)

We will see in §5.20 that the forgetful functor **Top** $\to$ **Set** lifts limits uniquely, but does not reflect limits (and hence doesn't create them). Also see §5.21.

5.20 5.3.11, p.140

(a) The formula of Example 5.1.22 (eq.5.16, p.121) is (with a slight change of notation)

$$\varprojlim D = \{(x_i)_{i \in \mathbf{I}} | x_i \in D_i \,\&\, \bar{u}(x_i) = x_j \, (u : i \to j)\}$$

(where of course i and u range over all objects and morphisms in $\mathbf{I}$ respectively, and I've written D_i for $D(i)$ and $\bar{u}$ for $D(u)$).

So it's the subset of the direct product $\prod_i D_i$ defined by the equations

$$\bar{u}(x_i) = x_j \quad \forall u : i \to j \tag{1}$$

Now suppose that D is a diagram in **Group**, so each D_i is a group and each $\bar{u}$ a homomorphism. We can still form the direct product $\prod_i D_i$, defining the operations pointwise, and look at the subset defined by eq.1. We need to check that this subset is a subgroup, that each of the projections are homomorphisms, and that this subgroup $\lim D$ has the required universal property.

For the first check, just note that the $\bar{u}$'s are homomorphisms. The second check is trivial. For the third, suppose $H \xrightarrow{h_i} D_i$ is a set of homomorphisms satisfying

$$\bar{u}(h_i(y)) = h_j(y) \quad \forall i \xrightarrow{u} j, \forall y \in H \tag{2}$$

Then

$$h : y \mapsto (h_i(y))_{i \in \mathbf{I}}$$

is a homomorphism from H into the direct product $\prod_i D_i$; since eq.2 implies eq.1, it maps into the subgroup $\lim D$. If we build the usual commutative diagram out of h, the h_i's, and the projections $\lim D \to D_i$, then it's trivial to check that diagram does commute, and moreover specifies the homomorphism $H \to \lim D$ uniquely. So $\lim D$ is indeed a limit.

The first bullet of Def.5.3.5 (p.139) decrees that this $\lim D$ is the *only* way to lift the limit cone in **Set** to a cone in **Group**: that if G is a group with

$UG = \prod_i UD_i$ and each projection $p_i : G \to D_i$ a homomorphism, then $G = \lim D$. Let $x, y \in G$, with $x = (x_i)_{i\in\mathbf{I}}$ and $y = (y_i)_{i\in\mathbf{I}}$. Then

$$(xy)_i = p_i(xy) = p_i(x)p_i(y) = x_i y_i$$

Likewise for inverses and the identity of G. So the operations are defined pointwise in G and $G = \lim D$ (not merely isomorphic to it).

(b) The argument of part (a) applies virtually without change. Instead of one operation[8], we have a collection, but we define all of them pointwise to get $\lim D$, and homomorphisms must respect all the operations.

(c) Let's extend the exercise by looking at **Top**, **Hausdorff**, **CptHff**, and the various forgetful functors among them.

First $U : \mathbf{Top} \to \mathbf{Set}$. Each D_i is a topological space, each $\bar{u}$ is a continuous map, and $U(\lim D)$ is the subset of the product $\prod UD_i$ satisfying eq.1. We give $\prod D_i$ the product topology and the subset the subspace topology, thus defining $\lim D$. (This replaces the "pointwise" aspect of (a) and (b).) What about our three checks? Trivially $\lim D$ is a topological space. All the projections $p_i : \lim D \to D_i$ are continuous (easy). Suppose $H \xrightarrow{h_i} D_i$ is a set of continuous maps satisfying eq.2. As before, there is one and only one way to make the required diagram commute in **Set**, namely with the function

$$h : y \mapsto (h_i(y))_{i\in\mathbf{I}}$$

Again we have h mapping H into $\lim D$ because eq.2 implies eq.1. Continuity of h falls out of the definitions of the subspace and product topologies. That is, a map $h : H \to \lim D$ is continuous iff the all the compositions $H \xrightarrow{h} \lim D \hookrightarrow \prod D_i \xrightarrow{p_i} D_i$ are continuous, but these are just the h_i's.

So $\lim D$ is a limit. Does it obey the decree of the first bullet of Def.5.3.5? Let's write $L = (\lim D \xrightarrow{p_i} D_i)_{i\in\mathbf{I}}$ for the limit cone, and let's suppose $L' =$

[8] Or three, if you want to treat inverses and the identity as "built-in" operations of the group.

$(T \xrightarrow{p'_i} D_i)_{i \in \mathbf{I}}$ is another cone on D with $UL' = UL$. That is, $UT = U(\lim D)$ and $Up_i = Up'_i$ for all i. U is faithful, so $p_i = p'_i$ for all i. Since $\lim D$ is a limit, we have a continuous $h : T \to \lim D$ with $p_i h = p_i$ for all i, i.e., the i-th component of $h(x)$ equals the i-th component of x for all i and all $x \in T$. So Uh is the identity.

But this doesn't rule out T having a finer topology than $\lim D$. To go to extremes, we could endow $U(\lim D)$ with the discrete topology and we'd get a perfectly good cone L' with $UL' = UL$. In short, U does not reflect limits, and so doesn't create them. (See §5.19.) However, if we insist that L' be a limit cone, then the (set-theoretic) identity between T and $\lim D$ is a homeomorphism. In the terminology of §5.19, U lifts limits uniquely.

As for the sequence of forgetful functors **CptHff** $\hookrightarrow$ **Hausdorff** $\hookrightarrow$ **Top**, each is a full subcategory of the next. So let's take the general situation, $\mathcal{A} \hookrightarrow \mathcal{B}$ with $\mathcal{A}$ a full subcategory of $\mathcal{B}$. First thing to note: if UL is a limit cone in $\mathcal{B}$, then L is a limit cone in $\mathcal{A}$. (All the objects and morphisms of L and UL are the same; the 'U' just identifies the surrounding category. Note why the fullness assumption matters.)

Next, if UL is a cone on UD and the vertex of UL belongs to $\mathcal{A}$, then the whole cone L belongs to $\mathcal{A}$ by fullness. Putting all these observations together, limits lift uniquely from $\mathcal{B}$ to $\mathcal{A}$, *provided* the vertices lift.

Limits in **Top** are subspaces of direct products. A direct product of Hausdorff spaces is Hausdorff; likewise a subspace. So we have unique lifting of limits from **Top** to **Hausdorff**. But there's a bit more: since the subspace is defined by the equations (1), limits in **Hausdorff** are closed subspaces. Direct products of compact spaces are compact, as are closed subspaces, so we have unique lifting of limits from **Hausdorff** to **CptHff**.

It's obvious that unique lifting is preserved by composition of functors, so we have unique lifting from **Set** to all three of our topological categories. But we have one last twist: the forgetful functors from **CptHff** reflect limits, and thus create limits. Proof: if L is a cone on D in **CptHff** and UL is a limit cone, then UL lifts to a limit cone L' on D. So there is a continuous map from the vertex of L to the vertex of L'. Set-theoretically this is the identity map. But by basic topology (see §1.12), any continuous map from a compact space to a Hausdorff space sends closed sets to closed sets, so the topologies of the two vertices are the same and the vertices are identical objects in **CptHff**.

In contrast, the forgetful functors from **Hausdorff** don't reflect limits, by the same argument we gave for **Top**.

5.21 5.3.12, p.140

A much weaker hypothesis about the functor F suffices. Suppose that if there is a limit cone on FD, then there is a limit cone that lifts to a limit cone on D (see §5.19). Briefly, if M is a limit cone on FD (in $\mathcal{B}$), then there is a limit cone FL on FD such that L is a limit cone on D (in $\mathcal{A}$).

Both the strict notion of "create limits" (Def.5.3.5, p.139) and the "more healthy and inclusive" (top of p.140). imply this hypothesis. But it also holds if F lifts limits (§5.19). For example, the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ lifts limits, but does not create limits in either sense, because it does not reflect limits (see §5.20).

Let $D : \mathbf{I} \to \mathcal{A}$ be a diagram of shape $\mathbf{I}$ in $\mathcal{A}$. Then $FD : \mathbf{I} \to \mathcal{B}$ is a diagram of shape $\mathbf{I}$ in $\mathcal{B}$, so it has a limit cone because $\mathcal{B}$ has limits. So there is a limit cone L on D by our hypothesis. So $\mathcal{A}$ has limits of shape $\mathbf{I}$.

Next, suppose that L is a limit cone in $\mathcal{A}$ on D. Then FD is a diagram in $\mathcal{B}$, so by our hypothesis there is a limit cone L' on D with FL' a limit cone on FD. Since L and L' are limit cones on D, they are isomorphic. (See Remark 5.1.20, p.119: limits are unique up to unique isomorphism.) Functors preserve isomorphisms, so FL and FL' are isomorphic. Since FL'

is a limit cone, FL is too. So F preserves limits.

Combining this result with §5.20, we conclude that **Group** and other algebraic categories are complete; likewise **Top**, **Hausdorff**, and **CptHff**. The forgetful functor **Group** $\to$ **Set** preserves limits; likewise in the chain

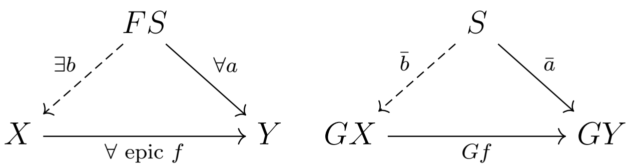
$$\mathbf{CptHff} \hookrightarrow \mathbf{Hausdorff} \hookrightarrow \mathbf{Top} \to \mathbf{Set}$$

the forgetful functors from any category to a later category preserves limits.

5.22 5.3.13, p.140

We spell out the meaning of projective: $P \in \mathcal{B}$ is projective iff for all epic $f : A \to B$, the map $a \mapsto fa$ is surjective. So any morphism from P to an object B can be factored through any epic to B.

(a) In the diagram below left, we need to show that for any $a : FS \to Y$ and any epic $f : X \to Y$, there exists a $b : FS \to X$ such that $fb = a$. We transfer over to **Set** using the adjunction G, getting the diagram below right.



It's safe to write $\bar{b}$ for the desired map from S to GX: since $F \dashv G$, every map $S \to GX$ is $\bar{b}$ for some $b : FS \to X$.

Since G preserves epics, Gf is epic. In **Set**, every object is projective: $\bar{b}$ must satisfy the equation $\bar{a}(s) = Gf(\bar{b}(s))$ for all $s \in S$, but since Gf is surjective, we can achieve this by choosing $\bar{b}(s) \in (Gf)^{-1}(\bar{a}(s))$.

Now we have to transfer back to $\mathcal{B}$. By naturality of adjunctions, $\overline{fb} = (Gf)\bar{b}$, which equals $\bar{a}$, so $fb = a$.

Note that the same proof works if we replace **Set** with any category where all objects are projective. Also note this corollary: free algebraic objects (free groups, free abelian groups, vector spaces over a field $k, \ldots$) are projective.

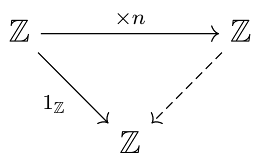
(b) In **Abelian**, the surjective homomorphisms are epic, since the forgetful functor to **Set** pulls back epics (item (8) of §1.11; the converse is easy to prove, but we won't need it.) So the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}_n$ is epic.

Let $P = \mathbb{Z}_n$, $A = \mathbb{Z}$, $B = \mathbb{Z}_n$. The only map $P \to A$, i.e., $\mathbb{Z}_n \to \mathbb{Z}$, is the zero map. So the identity $\mathbb{Z}_n \to \mathbb{Z}_n$ does not factor through the canonical epimorphism, and $\mathbb{Z}_n$ is not projective.

(c) We spell out the meaning of injective, with reference to the original category $\mathcal{B}$: I is injective iff any morphism $A \to I$ can be factored through any monic $A \to B$.

Suppose I is a vector space over k and $a : A \to I$ is a linear map and $f : A \to B$ is a monomorphism. By basic linear algebra, the image $f(A)$ has a complement in B, i.e., B can be written as a direct sum $B = f(A) \oplus N$. The range restriction of f, $f' : A \to f(A)$, is an isomorphism, so we can compose its inverse with a to get a map $b' : f(A) \to I$. Then $b'f' = a$. We extend b' to all of B by letting N be the null space: $b(y + n) = b'(y)$ for all $y \in f(A)$, $n \in N$. So $bf = a$.

In **Abelian**, $\mathbb{Z}$ is not injective, as we see from this diagram for any fixed $n > 1$:



6 Adjoints, representables and limits

6.1 6.1.5, p.146

We let $\mathbf{I} = \{1, 2\}$. A diagram $D : \mathbf{I} \to \mathcal{A}$ is just a pair of objects (D_1, D_2), so $[\mathbf{I}, \mathcal{A}] = \mathcal{A} \times \mathcal{A}$ and $\Delta A = (A, A)$ (as Leinster points out on p.143). A cone on D with vertex A, say $a : \Delta A \Rightarrow D$, is a pair of morphisms $a_i : A \to D_i$ $(i = 1, 2)$. (Although this a is a natural transformation, we prefer not to use a greek letter for it.) A limit is a cone $p : \Delta P \Rightarrow D$ (i.e., a vertex P and morphisms $p_i : P \to D_i$) such that for any cone a on D with vertex A, there is a unique morphism $f : A \to P$ such that $p_i f = a_i$ $(i = 1, 2)$. In other words, P is a product $D_1 \times D_2$ with projections p_i.

The functor $\text{Cone}(-, D) : \mathcal{A}^{\text{op}} \to \mathbf{Set}$ takes $A \in \mathcal{A}$ to the set of all cones on D with vertex A, i.e., all the pairs (a_1, a_2) with $a_i : A \to D_i$. This functor takes a morphism $f : A \to A'$ to the "morphism of cones" $(a'_1, a'_2) \mapsto (a'_1 f, a'_2 f) = (a_1, a_2)$. A representation of a contravariant functor $X : \mathcal{A}^{\text{op}} \to \mathbf{Set}$ sets up natural bijections $X(A) \leftrightarrow \mathcal{A}(A, P)$ for some $P \in \mathcal{A}$. So a representation of $\text{Cone}(-, D)$ should give us bijections $\text{Cone}(A, D) \leftrightarrow \mathcal{A}(A, P)$. By Coro.4.3.2, there is a "universal element" $p \in X(P) = \text{Cone}(P, D)$ such that for each $a \in \text{Cone}(A, D)$ there is a unique $f : A \to P$ with $Xf(p) = a$, i.e., $(p_1, p_2) \mapsto (p_1 f, p_2 f) = (a_1, a_2)$. So a representation is basically the same as a product $(p_i : D_1 \times D_2 \to D_i)_{i=1,2}$. This is Prop.6.1.1 for the case $\mathbf{I} = \{1, 2\}$.

Summarizing: we start with a diagram $D = (D_1, D_2)$ of shape $\{1, 2\}$, then look at all cones a with all vertices A over D; among these, we find a particular vertex P, and among the cones of $\text{Cone}(P, D)$ we find a particular cone (p_1, p_2). The universal element property of Coro.4.3.2 (p.99) is the same as the limit property of Prop.6.1.1 (p.143).

Coro.6.1.2 simply says here that products are unique up to isomorphism. Lemma 6.1.3(a) says that given two products $D_1 \times D_2$ and $D'_1 \times D'_2$ and morphisms $\alpha_i : D_i \to D'_i$ $(i = 1, 2)$, there is a unique $\alpha : D_1 \times D_2 \to D'_1 \times D'_2$ satisfying the obvious commutation condition $\alpha_i p_i = p'_i \alpha$. Lemma 6.1.3(b) adds that if we have morphisms $f_i : A \to D_i$ and $f'_i : A \to D'_i$, then any $s : A \to A'$ commuting with $\alpha_i : D_i \to D'_i$ $(i = 1, 2)$ will also commute with $\alpha : D_1 \times D_2 \to D'_1 \times D'_2$.

Prop.6.1.4 says that if $\mathcal{A}$ has all binary products, then $(D_1, D_2) \mapsto D_1 \times D_2$ is a functor from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$, which is right adjoint to the functor $A \mapsto (A, A)$. Leinster's last paragraph points out that we have to make an arbitrary choice of $D_1 \times D_2$ for each pair (D_1, D_2), but no big deal since all possible choices are isomorphic.

Finally, I note that the general case isn't all that different. We have $(D_i)_{i \in I}$ instead of $(D_i)_{i=1,2}$, and commutativity conditions flowing from the morphisms $Df : D_i \to D_j$ in the diagram, but these mostly take care of themselves.

6.2 6.1.6, p.146

Example 1.2.8 (p.22) tells us that a diagram $D : \mathbf{I} \to \mathbf{Set}$ is a (left) G-set. If $*$ stands for the single object in $\mathbf{I}$ treated as a category, then $D(*)$ is the underlying set of the G-set, and $a \cdot x = D(a)x$ is the action, where the morphism $a \in \mathbf{I}$ is an element of the group. Example 1.3.4 (p.29) tells us that a natural transformation $\alpha : D \Rightarrow D'$ is a G-equivariant map.

What does the diagonal functor Δ look like here? Given a set A, $\Delta(A)$ is the functor $\mathbf{I} \to A$ that sends $*$ to the set A, and every morphism $a \in \mathbf{I}$ to the identity function on A. In other words, $\Delta(A)$ is A with the trivial action. In §2.5 (Exercise 2.1.16(a), p.50), this was called Triv.

What is a cone on D? It's a set X and a function $f : X \to D(*)$ such that $D(a)f = f$ for every morphism a of $\mathbf{I}$. In other words, $a \cdot f(x) = f(x)$ for every a in the group and every $x \in X$, so the image of f is left pointwise fixed by the action of the group.

What is a limit cone on D? It's a cone $f : X \to D(*)$ such that any other cone factors through f; it's easy to see that this forces f to be a bijection between X and the fixed points of D, $\mathrm{Fix}(D)$.

We now see that Lemma 6.1.3(a) (p.144) says, in essence, that any G-equivariant map $D \to D'$ restricts to a map $\mathrm{Fix}(D) \to \mathrm{Fix}(D')$. Prop.6.1.4 (p.145) says that the functor $\mathrm{Fix} : \textbf{G-set} \to \textbf{Set}$ is right adjoint to $\Delta = \mathrm{Triv} : \textbf{Set} \to \textbf{G-set}$, which we saw in §2.5.

Now for the dual. A cocone is a function $f : D(*) \to X$ such that $f(a \cdot d) = f(x)$ for every a in the group and every $d \in D(*)$. So each orbit is sent to a single element of X. It's not hard to see that a colimit is (up to isomorphism) the function $d \mapsto \mathrm{orbit}(d)$, sending the G-set D to its set of orbits $\mathrm{Orb}(D)$. In §2.5 we saw that Orb is left adjoint to Triv.

So $\mathrm{Orb} \dashv \mathrm{Triv} \dashv \mathrm{Fix}$ is a special case of $\varinjlim \dashv \Delta \dashv \varprojlim$.

See also §2.5 and §6.9(c).

6.3 The Density Theorem

I found it difficult at first to wrap my mind around the proof of the Density Theorem (Theorem 6.2.17, pp.155–156). Herewith some thoughts that helped me feel comfortable with it.

First, why do we need the category of elements? Why not just use the Yoneda imbedding $H_\bullet : \mathbf{A} \to \mathrm{Presheaves}(\mathbf{A})$ as the diagram? But that diagram doesn't depend on the functor X at all, and so we can hardly expect its colimit to give us X.

This highlights the key difference between $H_\bullet$ and $H_\bullet \circ P$. An arbitrary

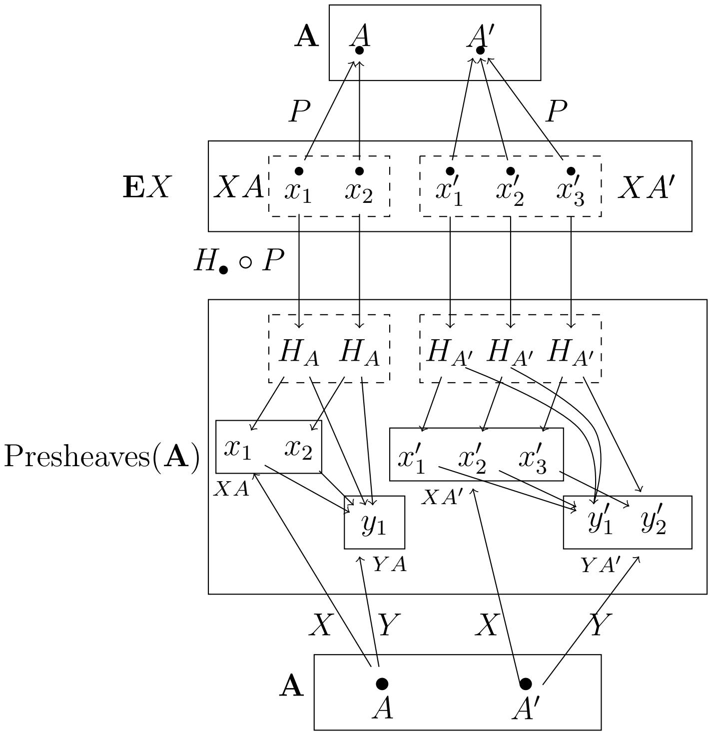


Figure 6: Density Theorem

presheaf X will typically be much "richer" than a representable H_A—often $H_A(A)$ contains only one element for any $A \in \mathbf{A}$, but XA typically contains *many* elements. (Example: the presheaf X of continuous functions on a topological space—see §4.12.) In the diagram $H_\bullet \circ P$, the H_A's appear "with multiplicity": we have as many "copies" of H_A as there are elements of XA.

Fig.6 illustrates this. At the top and the bottom of the figure we have the category $\mathbf{A}$, with the two objects indicated by labeled dots. The presheaf X has $XA = \{x_1, x_2\}$ and $XA' = \{x'_1, x'_2, x'_3\}$. The category of elements $\mathbf{E}X$ has five objects, namely the pairs (A, x_1), (A, x_2),..., (A', x'_3). In the figure I label these objects more succinctly, and use dashed boxes to group the objects according to the $\mathbf{A}$-object they come from. The projection P is shown going from $\mathbf{E}X$ to $\mathbf{A}$.

Most of the action takes place in the big middle box, Presheaves($\mathbf{A}$). The objects of this category are presheaves over $\mathbf{A}$, but I have adopted a non-uniform presentation. The diagram $H_\bullet \circ P$ consists of the representables H_A and $H_{A'}$, which I display "with multiplicity". That is, $H_\bullet \circ P$ sends both (A, x_1) and (A, x_2) to the same representable H_A, so I display it twice; likewise for the three $H_{A'}$'s.

Note that the "legs" of cocones (and cones) are indexed by the small index category: $(DI \xrightarrow{l_I} V)_{I \in \mathbf{I}}$ for a cocone with vertex V on a diagram $D : \mathbf{I} \to \mathcal{C}$. We can have $DI = DJ$ with $l_I \neq l_J$.

I indicate the functors X and Y by showing the sets XA, XA', YA, and YA' as boxes with their elements inside. Arrows from the bottom box to the middle box suggest the functors X and Y. The "guts" of X and Y are, so to speak, spread out inside the presheaf box.

Now let's think about the cocone with vertex X and diagram $H_\bullet \circ P$. For each object (A, x_i) in $\mathbf{E}X$, we need a leg from its image H_A to X—i.e., a morphism in Presheaves($\mathbf{A}$). In other words, an element of Presheaves($\mathbf{A}$)$[H_A, X]$.

The Yoneda lemma says there is a canonical bijection

$$\text{Presheaves}(\mathbf{A})[H_A, X] \cong XA$$

I indicate the leg (aka morphism aka natural transformation) with an arrow from the copy of H_A to the displayed element of XA.

Note the 1–1 correspondence between the dashed box of H_A's and the box XA, likewise for $H_{A'}$ and XA'. We have the multiplicity of H_A's and $H_{A'}$'s to thank for this (and the Yoneda lemma, of course). *That's* why we need the category of elements.

OK, what about the cocone with vertex Y? The Yoneda lemma says that Presheaves$(\mathbf{A})[H_A, Y] \cong YA$, so the requisite legs look like arrows from the H_A's and $H_{A'}$'s to elements in the boxes YA and YA'. But then we immediately get mappings from XA to YA, and from XA' to YA': just use the H_A's and $H_{A'}$'s as intermediaries (go up and then down). These mappings $XA \to YA$, $XA' \to YA'$ meld together to form a natural transformation $\overline{\alpha} : X \Rightarrow Y$.

To relate this to Leinster's proof: he notates the cocone on Y thus (p.156)

$$\left(H_A \xrightarrow{\alpha_{A,x}} Y\right)_{A \in \mathbf{A}, x \in XA}$$

and uses the Yoneda mapping $\alpha_{A,x} \mapsto y_{A,x}$ to define the functions $\overline{\alpha}_A : x \mapsto y$ for each A.

Several things need checking: the commutativity conditions on the cocones, the naturality of $\overline{\alpha}$, the commutativity conditions on $\overline{\alpha}$, and the uniqueness of $\overline{\alpha}$. Fig.7 will guide us through this thicket. I've removed some of the clutter of fig.6 to make room for different clutter. I will argue "the general from the particular": i.e., x_2 and x'_1 are intended as "typical" elements of XA and XA', etc.

First, I've used ξ_{A2} and $\xi_{A'1}$ to denote two of the morphisms from the cocone on $H_\bullet \circ P$ with vertex X. (Leinster would denote these ξ_{A,x_2} and ξ_{A',x'_1}.) Two

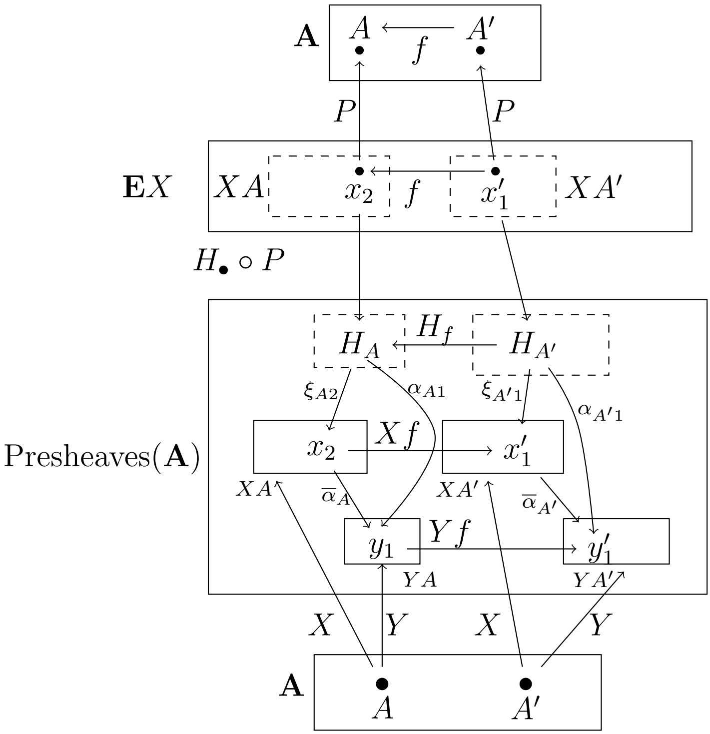


Figure 7: Commutativity Conditions

layers are implicit here: ξ_{A2} is a natural transformation $H_A \Rightarrow X$, whose components are therefore functions $(\xi_{A2})_B : H_A(B) \to XB$. Fortunately the Yoneda lemma tells us that ξ_{A2} is completely determined by the single element $x_2 \in XA$, so we get away with only a single arrow in the figure.

I've used α_{A1} and $\alpha_{A'1}$ to denote two of the morphisms from the cocone on $H_\bullet \circ P$ with vertex Y. Also, $\overline{\alpha}_A$ and $\overline{\alpha}_{A'}$ denote two components of the natural transformation $\overline{\alpha} : X \Rightarrow Y$.

We will need two results (both on p.97) from the proof of the Yoneda lemma. First, the fact that a natural transformation $\tau : H_A \Rightarrow X$ is determined by its single value $\tau(1_A) \in XA$. Second, Eq.4.5 that justifies this. With slightly tweaked notation:

$$\tau_{A'}(f) = (Xf)\tau_A(1_A), \qquad (f : A' \to A) \tag{3}$$

To keep fig.6 simple, I left out any morphisms in $\mathbf{E}X$, but of course the things we have to check arise from just these morphisms. Fig.7 displays a morphism $f : (A', x_1') \to (A, x_2)$. By definition of $\mathbf{E}X$, this is an $f : A' \to A$ in $\mathbf{A}$ such that $(Xf)x_2 = x_1'$. Note that $Pf = f$, and recall that $H_\bullet : f \mapsto H_f$, where H_f is defined by

$$\begin{aligned} H_f &: H_{A'} \Rightarrow H_A, & f &: A' \to A \\ H_f &: H_{A'}(B) \to H_A(B), & B &\in \mathbf{A} \\ H_f &: a \mapsto fa, & a &: B \to A' \end{aligned}$$

(The fussy reader will complain that two of those H_f's should have been $(H_f)_B$.) The natural transformations ξ_{Ai} were defined via the Yoneda lemma, so we have to verify that they form a cocone. The equation to check for f is

$$\xi_{A2} H_f = \xi_{A'1} \tag{4}$$

ξ_{A2} and $\xi_{A'1}$ are determined by the requirements

$$(\xi_{A2})_A(1_A) = x_2 \in XA, \qquad (\xi_{A'1})_{A'}(1_{A'}) = x_1' \in XA'$$

To verify (4), it's enough to evaluate both sides at $1_{A'}$. We use (3) with $\tau = \xi_{A2}$, $B = A'$, and $a = 1_{A'}$.

$$\begin{aligned}
(\xi_{A2})_{A'} H_f(1_{A'}) &\stackrel{?}{=} (\xi_{A'1})_{A'} 1_{A'} \\
(\xi_{A2})_{A'}(f) &\stackrel{?}{=} x_1' \\
(Xf)(\xi_{A2})_A(1_A) &\stackrel{?}{=} x_1' \\
(Xf)x_2 &\stackrel{\checkmark}{=} x_1'
\end{aligned}$$

So the ξ's form the legs of a cocone.

The α's are *assumed* to form the legs of a cocone with vertex Y. I.e.,

$$(\alpha_{A1})_{A'} H_f(1_{A'}) = (\alpha_{A'1})_{A'} 1_{A'}$$

and evaluating both sides just like for ξ, we conclude

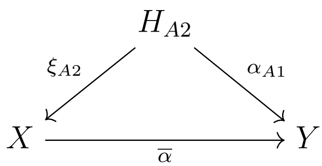
$$(Yf)y_1 = y_1'$$

Now, $\overline{\alpha}_A(x_2) = y_1$ and $\overline{\alpha}_{A'}(x_1') = y_1'$. If you stare at the Presheaves box in fig.7 for a moment, you'll see this naturality square:

$$\begin{array}{ccc}
XA & \xrightarrow{Xf} & XA' \\
{\scriptstyle \overline{\alpha}_A}\downarrow & & \downarrow{\scriptstyle \overline{\alpha}_{A'}} \\
YA & \xrightarrow[Yf]{} & YA'
\end{array}$$

So $\overline{\alpha}$ is a natural transformation.

Finally, we do the commutativity condition and uniqueness of $\overline{\alpha}$ together. Let's write H_{A2} for the "copy" of H_A associated with the leg ξ_{A2}. The equation $\overline{\alpha}\xi_{A2} = \alpha_{A1}$, stemming from $(A, x_2) \in \mathbf{E}X$, looks like this:

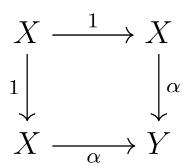


The diagram stands for an equation among natural transformations, which holds if and only if it holds when evaluated at $1_A \in H_A(A)$. That evaluation reduces to $\overline{\alpha}_A(x_2) = y_1$, and the "go up then down" prescription given for $\overline{\alpha}$ insures this. So commutativity holds, but commutativity also prescribes the value of $\overline{\alpha}_A(x_2)$ for every $(A, x_2) \in \mathbf{E}X$, making $\overline{\alpha}$ unique.

6.4 6.2.20, p.157

(a) In one direction this is easy. Suppose α_A monic for all $A \in \mathcal{A}$. Let $\alpha\beta = \alpha\beta'$ for two natural transformations $\beta, \beta' : W \Rightarrow X$. This means that $\alpha_A\beta_A = \alpha_A\beta'_A$ for all A, and so $\beta_A = \beta'_A$ for all A, i.e., $\beta = \beta'$.

In the other direction, suppose α is monic in $[\mathbf{A}, \mathcal{S}]$. By Lemma 5.1.32 (p.124) the following diagram is a pullback square:



By Coro. 6.2.6 (p.151), the evaluation functor ev_A preserves pullbacks for each $A \in \mathcal{A}$. Here we use the hypothesis that $\mathcal{S}$ has pullbacks (see Warning 6.27, p.151), i.e., all limits of shape $\mathbf{P}$; $\mathbf{P}$ is defined in eq.5.14 (p.118), and looks like a pullback square with the northwest corner missing. Applying

ev_A to the pullback square for α gives

$$\begin{array}{ccc} XA & \xrightarrow{1} & XA \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \alpha_A} \\ X & \xrightarrow[\alpha_A]{} & YA \end{array}$$

So α_A is monic by Lemma 5.1.32.

(b) By part (a), α is monic iff α_A is injective for all A, and α is epic iff α_A is surjective for all A. (Note that we don't switch monic and epic: the assertion of (a) holds for all small categories $\mathbf{A}$, and so in particular for the small category $\mathbf{A}^{\mathrm{op}}$. Observe also that $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \neq [\mathbf{A}, \mathbf{Set}]^{\mathrm{op}}$.)

(c) The first direction in part (a) did not use the fact about pointwise limits. It remains to prove the other direction, in contrapositive form. We need to show that if $\alpha : X \Rightarrow Y$ has a non-injective α_A, then α is not monic. So we need $\beta, \beta' : W \Rightarrow X$ such that $\alpha\beta = \alpha\beta'$ but $\beta \neq \beta'$.

Since $\alpha_A : XA \to YA$ is not injective, we have $\alpha_A(a) = \alpha_A(a')$ for some $a \neq a'$, $a, a' \in XA$. Let $W = H_A$. By the Yoneda lemma, natural transformations $H_A \Rightarrow X$ are in 1–1 correspondence with elements of XA, with $\beta : H_A \Rightarrow X$ uniquely determined by the value of $\beta_A(1_A)$. Let $\beta_A(1_A) = a$ and $\beta'_A(1_A) = a'$. So $\beta \neq \beta'$. However, the compositions $\alpha\beta$ and $\alpha\beta'$ are natural transformations $H_A \Rightarrow Y$ agreeing at 1_A, and so $\alpha\beta = \alpha\beta'$.

For epics, just dualize.

6.5 6.2.21, p.157

(a) First we elucidate the sum of presheaves $X+Y$. (It's dual to $X \times Y$, considered at the top of p.150.) Two equations tell the story:

$$(X+Y)A = XA + YA$$
$$(X+Y)f = Xf + Yf$$

Recall that the direct sum of functions $Xf : XA \to XB$ and $Yf : YA \to YB$ is the function defined by cases on domain $XA + YA$:

$$(Xf+Yf)a = \begin{cases} (Xf)a \text{ when } a \in XA \\ (Yf)a \text{ when } a \in YA \end{cases}$$

It's easy to verify directly the compliance of this construction with the colimit definition of the sum. Alternately, we can appeal to the dual of Theorem 6.2.5 (p.149)—colimits are computed pointwise.

Assume that $H_A \cong X+Y$, so we have a natural equivalence $\alpha : H_A \Leftrightarrow X+Y$. This means that for each B we have a bijection between $H_A(B)$ and $XB+YB$. In other words, we have a partition of $H_A(B)$ for each B; let's write $H_A(B) = \overline{XB} + \overline{YB}$. Moreover, the naturality squares mean that $H_A(f)$ respects the partitions: if $f : B \to A$, then $H_A(f)$ sends $\overline{XA}$ into $\overline{XB}$ and $\overline{YA}$ into $\overline{YB}$.

Now we look at $1_A \in H_A(A) = XA+YA$. Suppose $1_A \in \overline{XA}$. If $f \in H_A(B)$, i.e., $f : B \to A$, then $H_A(f) : 1_A \to f$. Since $H_A(f)$ respects the partitions, it follows that $f \in \overline{XB}$. I.e., every element of $H_A(B)$ belongs to $\overline{XB}$, so all $\overline{YB} = \varnothing$. It follows that all $YB = \varnothing$ as well. If $1_A \in \overline{YA}$ then all $XB = \varnothing$ by the same reasoning.

A slight variant on this proof: work directly with the partitions $XA+YA$, instead of transferring them over to H_A. Because $X+Y$ is equivalent to H_A, we can appeal to Coro.4.3.2 (p.99). The universal element u takes the place of 1_A, and the equation $(X\bar{x})u = x$ replaces $H_A(f) : 1_A \to f$. If u belongs to XA then all $YB = \varnothing$, etc.

(b) No representable can ever equal the identically $\varnothing$ presheaf, since $H_A(A)$ always contains at least 1_A.

6.6 6.2.22, p.158

Let 1 be a singleton in **Set** and let X be a presheaf on $\mathbf{A}$. Consider $(1 \Rightarrow X)$. Its objects are pairs $(A, 1 \xrightarrow{x} XA)$ with $A \in \mathbf{A}^{\text{op}}$. Since $XA \in \mathbf{Set}$, $1 \xrightarrow{x} XA$ can be identified with an element of XA. A morphism in $(1 \Rightarrow X)$ from $(A', 1 \xrightarrow{x'} XA')$ to $(A, 1 \xrightarrow{x} XA)$ is a morphism $f : A' \to A$ in $\mathbf{A}^{\text{op}}$ such that this diagram commutes:

$$\begin{array}{ccc} 1 & \xrightarrow{x} & XA \\ & {\scriptstyle x'}\searrow & \downarrow{\scriptstyle Xf} \\ & & XA' \end{array}$$

which is exactly the condition that $(Xf)x = x'$.

6.7 6.2.23, p.158

The category of elements of a presheaf $X : \mathcal{A}^{\text{op}} \to \mathbf{Set}$ consists of pairs $(A \in \mathcal{A}, u \in XA)$; morphisms $(B, x) \xrightarrow{f} (A, u)$ are morphisms $B \xrightarrow{f} A$ such that $(Xf)u = x$ (Def.6.2.16, p.155). So (A, u) is terminal iff for every $x \in XB$ there is a unique $f : B \to A$ such that $(Xf)u = x$. But this is the universal element criterion for a presheaf to be representable (Coro.4.3.2, p.99).

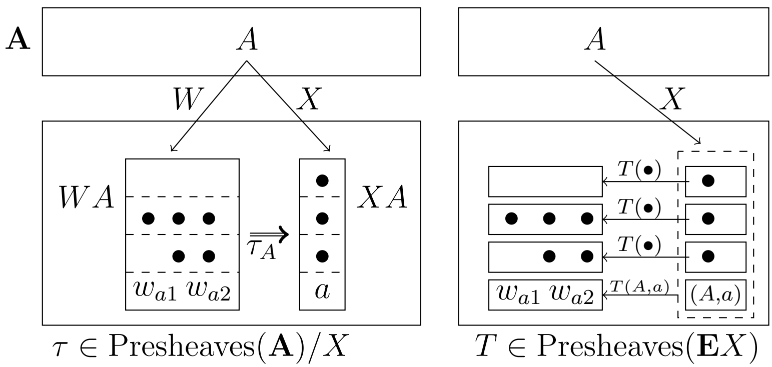


Figure 8: Slice of a Presheaf

6.8 6.2.24, p.158

We will show that $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$ is equivalent to $[\mathbf{E}X^{\mathrm{op}}, \mathbf{Set}]$, where $\mathbf{E}X$ is the category of elements (Def.6.2.16, p.155). For intuition, consult fig.8. (I write Presheaves($\mathbf{C}$) for the category $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$.)

In the left half of the figure, we see an object τ of the slice category Presheaves($\mathbf{A}$)$/X$. Recall that the objects of Presheaves($\mathbf{A}$)$/X$ are morphisms of Presheaves($\mathbf{A}$), which are natural transformations $\tau : W \Rightarrow X$. So for any $A \in \mathbf{A}$, τ_A is a function from WA to XA. In the figure, τ_A is given by horizontal projection. Note that the illustrated τ_A is neither onto (the top element of XA has no preimage), nor 1–1. The dashed lines emphasize how τ_A partitions WA, each partition being an inverse image $\tau_A^{-1}(a)$ for some $a \in XA$.

In the right half of the figure, we see an object of the presheaf category Presheaves($\mathbf{E}X$). Recall that the objects of Presheaves($\mathbf{E}X$) are contravariant functors T from $\mathbf{E}X$ to $\mathbf{Set}$, and the objects in $\mathbf{E}X$ are pairs (A, a) with $a \in XA$. So for any $(A, a) \in \mathbf{E}X$, $T(A, a)$ is a set. The dashed box groups all the objects (A, a) that come from the same $A \in \mathbf{A}$.

To show that our categories are equivalent, we need functors P and Q from left to right and right to left. That is, if τ and T correspond (up to natural equivalence):

$$\begin{aligned}
&P : \text{Presheaves}(\mathbf{A})/X \to \text{Presheaves}(\mathbf{E}X)\\
&P : \tau \mapsto P\tau \cong T\\
&Q : \text{Presheaves}(\mathbf{E}X) \to \text{Presheaves}(\mathbf{A})/X\\
&Q : T \mapsto QT \cong \tau
\end{aligned}$$

The actions of P and Q on morphisms also need defining, but we'll address this later. We also need natural equivalences $QP \cong 1_{\text{left}}$ and $PQ \cong 1_{\text{right}}$.

The definitions of $P\tau$ and QT can almost be read off the figure. For each $(A, a) \in \mathbf{E}X$, $P\tau(A, a)$ is a set, so let

$$P\tau(A, a) = \tau_A^{-1}(a)$$

$P\tau$ is a functor, so we need to define $P\tau(f)$ for any morphism $(B, b) \xrightarrow{f} (A, a)$ in $\mathbf{E}X$. $P\tau(f)$ will be a function from the set $P\tau(A, a)$ to the set $P\tau(A, a)$. Because τ is a natural transformation, we have the commuting square

$$\begin{array}{ccc}
WA & \xrightarrow{Wf} & WB \\
\downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_B} \\
XA & \xrightarrow[Xf]{} & XB
\end{array}$$

So

$$\begin{aligned}
\tau_B(Wf) &= (Xf)\tau_A\\
\tau_B^{-1}\tau_B(Wf)\tau_A^{-1}(a) &= \tau_B^{-1}(Xf)\tau_A\tau_A^{-1}(a)\\
(Wf)\tau_A^{-1}(a) &\subseteq \tau_B^{-1}(b)\\
(Wf)P\tau(A, a) &\subseteq P\tau(B, b)
\end{aligned}$$

(Recall that if $(B,b) \xrightarrow{f} (A,a)$, then $(Xf)a = b$. We've also used the set-theoretic inequality $K \subseteq h^{-1}hK$.) So if we restrict Wf to $P\tau(A,a)$, it maps into $P\tau(B,b)$. Define

$$P\tau(f) = Wf|_{P\tau(A,a)}$$

I leave it to you to check the functoriality of $P\tau$ (i.e., $P\tau(fg) = P\tau(f)P\tau(g)$).

Defining QT involves more moving parts. QT is a natural transformation $W \Rightarrow X$, so we need to define both the (contravariant) functor W (let's denote it W_T) and the functions QT_A for each $A \in \mathbf{A}$. $W_T A$ and $W_T f$ must be specified for each A and each morphism $f : B \to A$ in $\mathbf{A}$.

$$\begin{aligned} W_T A &= \{(w,a) | a \in XA, w \in T(A,a)\} \\ W_T f &: (w,a) \mapsto ((Tf)w, (Xf)a) \qquad f : (B,b) \to (A,a) \\ QT_A &: W_T A \to XA \\ QT_A &: (w,a) \mapsto a \end{aligned}$$

Let's examine these formulas, starting with the definition of $W_T A$. The figure suggests collecting all the w_{ai}'s as a ranges over XA, i.e., the union $\bigcup_{a \in XA} T(A,a)$. However, the sets $T(A,a)$ might not be disjoint. By contrast, as we saw, the different "strata" of WA—the inverse image sets $\tau_A^{-1}(a)$—partition WA. So we tag each w_{ai} with its a before combining the $T(A,a)$'s. Effectively, we form the disjoint union $\bigsqcup_{a \in XA} T(A,a)$.

To understand the formula for $W_T f$, we need to delve into the artful language Leinster uses to define morphisms in $\mathbf{E}X$. He says (conforming his notation to ours) that "maps $(B,b) \to (A,a)$ are maps $f : B \to A$ in $\mathbf{A}$ such that $(Xf)a = b$" (p.155). Now, a morphism in $\mathbf{A}$ can't literally be a morphism in $\mathbf{E}X$. Rather, a morphism in $\mathbf{A}$ *splits* into multiple morphisms in $\mathbf{E}X$. For each pair $(a,b) \in XA \times XB$ with $(Xf)a = b$, we have a morphism $(B,b) \to (A,a)$.[9] So the morphism $(B,b) \to (A,a)$ is uniquely

[9]Visually, the single arrow $Xf : XA \to XB$ splits into all the arrows from the individual elements of XA to their targets in XB. The direction of the arrows in $\mathbf{E}X$ is reversed, however, because X is contravariant.

identified by f in $\mathbf{A}$ and $a \in XA$. Writing f_a for this morphism, we have $Tf_a : T(A,a) \to T(B,b)$. We can also write, more precisely:

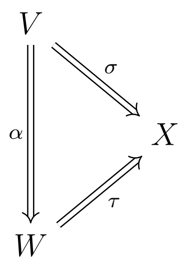
$$W_T f : (w,a) \mapsto ((Tf_a)w, (Xf)a)$$

Essentially, $W_T f$ is defined one stratum at a time, cobbling together the Tf_a's for all the a's in XA.[10]

The functoriality of W_T follows from the functoriality of X and of T.

The definition of QT_A is simplicity itself, just horizontal projection in the figure. The naturality of QT follows routinely from the definitions.

Next, the action of P on morphisms in Presheaves$(\mathbf{A})/X$, i.e., $\alpha : V \Rightarrow W$ making this diagram commute:



We have $\sigma = \tau\alpha$, and hence for each A and each $a \in XA$

$$\begin{aligned}
\sigma_A &= \tau_A \alpha_A \\
\sigma_A^{-1} &= \alpha_A^{-1} \tau_A^{-1} \\
\sigma_A^{-1}(a) &= \alpha_A^{-1} \tau_A^{-1}(a) \\
\alpha_A \sigma_A^{-1}(a) &\subseteq \tau_A^{-1}(a) \\
\alpha_A P\sigma(A,a) &\subseteq P\tau(A,a)
\end{aligned}$$

[10] Note that if $T(A,a)$ and $T(A,a')$ intersect, say containing a common element w, there is no guarantee that $Tf_a(w) = Tf_{a'}(w)$. So it's a good thing we defined $W_T A$ using the disjoint union.

So we can define $P\alpha$ one stratum at a time, i.e.,

$$\begin{aligned} P\alpha_{(A,a)} &: P\sigma(A,a) \to P\tau(A,a) \\ P\alpha_{(A,a)} &: w \mapsto \alpha_A(w) \end{aligned}$$

(Note that $P\alpha : P\sigma \Rightarrow P\tau$. I leave it to you to check the naturality of $P\alpha$. Also left to you: the functoriality of P.)

Next stop: the action of Q on morphisms in Presheaves($\mathbf{E}X$). Suppose $\alpha : S \Rightarrow T$ is a natural transformation (i.e., a morphism in Presheaves($\mathbf{E}X$)). We will define a natural transformation $Q\alpha$ making this diagram commute:

$$\begin{array}{ccc} W_S & & \\ & \overset{QS}{\searrow} & \\ Q\alpha \Downarrow & & X \\ & \underset{QT}{\nearrow} & \\ W_T & & \end{array}$$

So $Q\alpha_A$ is a function from $W_S A$ to $W_T A$. We define it by

$$Q\alpha_A : (w, a) \mapsto (\alpha_{(A,a)}(w), a)$$

Again we do things stratum-by-stratum: (w, a) belongs to the a-stratum of $W_S A$, derived from $S(A, a)$. The component $\alpha_{(A,a)}$ of the natural transformation α maps $S(A, a)$ into $T(A, a)$. The tags (i.e., strata) don't change. To verify that $Q\alpha$ is natural, we check that this square commutes for any $f : B \to A$:

$$\begin{array}{ccc} W_S A & \xrightarrow{W_S f} & W_S B \\ {\scriptstyle Q\alpha_A}\downarrow & & \downarrow{\scriptstyle Q\alpha_B} \\ W_T A & \xrightarrow[W_T f]{} & W_T B \end{array}$$

This is just the stratified version of the naturality squares for α, S, and T

$$\begin{array}{ccc} S(A,a) & \xrightarrow{Sf_a} & S(B,b) \\ \Big\downarrow{\scriptstyle \alpha_{(A,a)}} & & \Big\downarrow{\scriptstyle \alpha_{(B,b)}} \\ T(A,a) & \xrightarrow[Tf_a]{} & T(B,b) \end{array}$$

aggregated over all $a \in A$.

After all this work, the punchline—the natural equivalences $QP \cong 1_{\text{left}}$ and $PQ \cong 1_{\text{right}}$—come as a bit of an anticlimax. PQT is this functor:

$$PQT(A,a) = \{(w,a)|w \in T(A,a)\}$$
$$PQTf(w,a) = (Tf(w), f(a))$$

i.e., it just tags all the elements of $T(A,a)$ with a, and does "tagged" version of Tf. So we have the obvious natural equivalence $\varphi : 1_{\text{right}} \Rightarrow PQ$:

$$\varphi_{(A,a)} : T(A,a) \to PQT(A,a)$$
$$\varphi_{(A,a)} : w \mapsto (w,a)$$

For QP, the story is pretty much the same. If $\tau : W \Rightarrow X$, then $P\tau$ separates the strata of WA, with w going into stratum $\tau_A(w)$. Q then tags w with $\tau_A(w)$ and aggregates the tagged elements. So:

$$\tau : W \Rightarrow X$$
$$QP\tau : W_{P\tau} \Rightarrow X$$
$$QP\tau_A : W_{P\tau}A \to XA$$
$$QP\tau_A : (w, \tau_A(w)) \mapsto \tau_A(w)$$

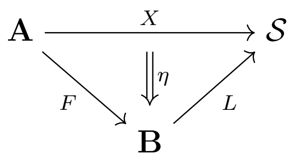
So the natural equivalence $\psi : 1_{\text{left}} \Rightarrow QP$ is just

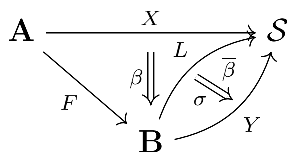
$$\psi_A : WA \to W_{P\tau}A$$
$$\psi_A : w \mapsto (w, \tau_A(w))$$

QED

6.9 6.2.25, p.158

(a) We will write L for $\mathrm{Lan}_F(X)$. The diagrams below illustrate the universal property of L; we explicate η, β, $\overline{\beta}$, and σ below.





Here $\eta : X \Rightarrow LF$ is a natural transformation, yet to be defined. For any natural transformation $\beta : X \Rightarrow YF$, we will define a natural transformation $\overline{\beta} : L \Rightarrow Y$. The desired bijection between natural transformations $\beta : X \Rightarrow YF$ and $\sigma : L \Rightarrow Y$ will be given by

$$\beta \mapsto \overline{\beta}, \qquad \sigma \mapsto (\sigma F)\eta$$

and we will show that these are inverse, i.e.,

$$(\overline{\beta}F)\eta = \beta, \qquad \overline{(\sigma F)\eta} = \sigma$$

(Note that $(\overline{\beta}F)\eta$ and $(\sigma F)\eta$ involve both horizontal and vertical composition.) Our task splits into these subtasks:

1. Define LB for all $B \in \mathbf{B}$.
2. Define Lb for all $b : B \to B'$.
3. Show that L is functorial.
4. Define $\eta_A : XA \to LFA$ for all $A \in \mathbf{A}$, and show that this defines a natural transformation $\eta : X \Rightarrow LF$.
5. Given any natural transformation $\beta : X \Rightarrow YF$, define $\overline{\beta}_B : LB \to YB$ for all $B \in \mathbf{B}$.

6. Show that $\overline{\beta}$ is a natural transformation $L \Rightarrow Y$.
7. Show that $(\overline{\beta}F)\eta = \beta$ and $\overline{(\sigma F)\eta} = \sigma$ for any $\beta : X \Rightarrow YF$ and any $\sigma : L \Rightarrow Y$.

(a1): Leinster defines LB as the colimit of the diagram

$$(F \Rightarrow B) \xrightarrow{P_B} \mathbf{A} \xrightarrow{X} \mathcal{S}$$

We dissect this, so as to fix some notation. Given $B \in \mathbf{B}$ we first find all $A_i \in \mathbf{A}$ such that there are morphisms $f_i : FA_i \to B$, and also look at all $a : A_i \to A_j$ satisfying the "triangle condition":

$$f_i = f_j(Fa)$$

This is $(F \Rightarrow B)$, with the f_i's corresponding to its objects and the triangle conditions to its morphisms. Next we form the diagram D_B in $\mathcal{S}$ with nodes XA_i and conditions $Xa : XA_i \to XA_j$, and let LB be the colimit of D_B. So there are legs $l_i : XA_i \to LB$ satisfying triangle conditions

$$l_i = l_j(Xa)$$

We let I be the index set for the f_i's, A_i's, and l_i's, so $(f_i)_{i\in I}$ is the family of morphisms $f_i : FA_i \to B$ and $(l_i)_{i\in I}$ is the family of legs $l_i : XA_i \to LB$. So basically:

$$\begin{aligned} (F \Rightarrow B) : \quad & (f_i : FA_i \to B)_{i\in I} \\ \text{Colimit cocone} : \quad & (l_i : XA_i \to LB)_{i\in I} \end{aligned}$$

Note that the f_i's are all distinct (indeed, in 1–1 correspondence with I), but the A_i's may not be. ("Multiplicity": see the discussion in §6.3.) Also, we should say "*a* colimit" rather than "*the* colimit". We must chose a colimit cocone (call it C_B) for each B. The mapping $f_i \mapsto l_i$ is the key to the construction.[11]

[11] Why key? Ignoring details, because it moves morphisms from $\mathbf{B}$ to $\mathcal{S}$, the chief accomplishment of L.

(a2): Suppose $b : B \to B'$. So we have a family

$$(f'_{i'} : FA'_{i'} \to B')_{i' \in I'}$$

satisfying a set of triangle conditions, and a colimit cocone

$$(l'_{i'} : XA'_{i'} \to LB')_{i' \in I'}$$

Since $f_i : FA_i \to B$, we have $FA_i \xrightarrow{f_i} B \xrightarrow{b} B'$, and so $bf_i : FA_i \to B'$ is one of the $f'_{i'}$'s sending $FA'_{i'}$ to B'. Here $A'_{i'} = A_i$. Set $f'_i = bf_i$, so we have a map

$$\begin{aligned}&(F \Rightarrow B) \to (F \Rightarrow B')\\&f_i \mapsto bf_i = f'_i\end{aligned}$$

and therefore $(F \Rightarrow B)$ maps into $(F \Rightarrow B')$. It's easiest to picture the case where $f_i \mapsto f'_i$ is injective. Then we can say that $I \subseteq I'$, A_i is one of the $A'_{i'}$'s, and f'_i is one of the $f'_{i'}$'s. In general we just have a map $I \to I'$, but the rest of the previous sentence holds. (About notation: i' ranges over I', i ranges over I, and f'_i, like $f'_{i'}$, belongs to $(F \Rightarrow B')$.)

Each triangle condition of $(F \Rightarrow B)$ maps to one for $(F \Rightarrow B')$: given $a : A_i \to A_j$ with $f_i = f_j(Fa)$, we have $bf_i = bf_j(Fa)$, i.e., $f'_i = f'_j(Fa)$. So the mapping $(F \Rightarrow B)$ into $(F \Rightarrow B')$ is a functor. It follows that D_B maps into a "subdiagram" of $D_{B'}$: each node XA_i in D_B appears in $D_{B'}$, and any $Xa : XA_i \to XA_j$ belonging to D_B also belongs to $D_{B'}$. So the mapping $I \to I'$ determines a "sub-cocone" of $C_{B'}$, i.e., a cocone on D_B with legs l'_i and vertex LB'.

Since C_B is a colimit cocone, there is a unique $Lb : LB \to LB'$ satisfying

$$l'_i = (Lb)l_i \qquad \forall i \in I$$

(a3) Suppose we have $B \xrightarrow{b} B' \xrightarrow{b'} B''$. The defining property of $L(b'b)$ is

$$l''_i = L(b'b)l_i \qquad \forall i \in I$$

However,

$$l_i'' = (Lb')l_i' \text{ and } l_i' = (Lb)l_i \qquad \forall i \in I$$

so

$$l_i'' = (Lb')(Lb)l_i \qquad \forall i \in I$$

so $L(b'b) = (Lb')(Lb)$.

Fine point: we have to show that the composition of the mappings $I \to I' \to I''$ is the mapping $I \to I''$. This follows from the equations $f_i{}'' = b'f_i' = b'bf_i$.

(a4) For any $A \in \mathbf{A}$, we have $1_{FA} : FA \to FA$ as a component of the diagram D_{FA}. In other words, there is an index $i \in I$ with $A_i = A$ and $f_i = 1_{FA}$; let's write 1 for that index. So $A_1 = A$, $f_1 = 1_{FA}$, and $XA_1 = XA$ is a node in D_{FA}. Hence we have a leg $l_1 : XA_1 \to LFA$. Set $\eta_A = l_1$, so $\eta_A : XA \to LFA$.

Now let's say $p : A \to A'$. We have to check the naturality square

$$\begin{array}{ccc} XA & \xrightarrow{Xp} & XA' \\ {\scriptstyle l_1=\eta_A}\downarrow & & \downarrow{\scriptstyle \eta_{A'}=l'_{1'}} \\ LFA & \xrightarrow[LFp]{} & LFA' \end{array}$$

From the defining property of LFp, we have

$$l_1' = (LFp)l_1$$

where l_1' is the leg in $C_{FA'}$ corresponding to $f_1' = (Fp)f_1 = (Fp)(1_{FA}) = Fp$.

Next observe that $Xp : XA \to XA'$ belongs to $D_{FA'}$, because the morphism $p : A \to A'$ satisfies the triangle condition

$$\begin{array}{ccc} FA & \xrightarrow{\quad Fp \quad} & FA' \\ & {\scriptstyle Fp}\searrow \quad \swarrow{\scriptstyle 1_{FA'}} & \\ & FA' & \end{array}$$

giving rise to the Xp arrow in $D_{FA'}$, and therefore a triangle condition in $C_{FA'}$:

$$\begin{array}{ccc} XA & \xrightarrow{Xp} & XA' \\ & {\scriptstyle l'_1}\searrow \quad \swarrow{\scriptstyle l'_{1'}} & \\ & LFA' & \end{array}$$

We justify the labels on the downward arrows: we noted above that l'_1 is the leg in $C_{FA'}$ corresponding to Fp, and of course $1'$ is the index in I' for which $A'_{1'} = A'$ and $f'_{1'} = 1_{FA'}$.

So $l'_1 = l'_{1'}(Xp)$. Combined with our earlier equation $l'_1 = (LFp)l_1$, this shows that the naturality square commutes.

(a5) We have (by hypothesis) a functor $Y : \mathbf{B} \to \mathcal{S}$ and a natural transformation $\beta : X \Rightarrow YF$; we want to define $\overline{\beta}_B$ for all $B \in \mathbf{B}$, with $\overline{\beta}_B : LB \to YB$. Our only handle on LB is through the colimit cocone C_B, so our only way forward is to construct a cocone on D_B with vertex YB. Then $\overline{\beta}_B$ will be the unique completing morphism between the cocone vertices.

We assemble the cocone out of two ingredients. The first is the set of morphisms $f_i : FA_i \to B$ of $(F \Rightarrow B)$; this gives us

$$Yf_i : YFA_i \to YB \qquad \forall i \in I$$

The second is the set of morphisms β_{A_i}, for which

$$\beta_{A_i} : XA_i \to YFA_i \qquad \forall i \in I$$

Composing these, we define

$$y_i = (Yf_i)\beta_{A_i} : XA_i \to YB$$

as desired. To show we have a cocone, we have to check the triangle conditions. These follow from the diagram

$$
\begin{array}{ccccc}
XA_i & & \xrightarrow{\quad Xa \quad} & & XA_j \\
\downarrow{\scriptstyle \beta_{A_i}} & & & & \downarrow{\scriptstyle \beta_{A_j}} \\
YFA_i & & \xrightarrow{\quad YFa \quad} & & YFA_j \\
& {\scriptstyle Yf_i}\searrow & & \swarrow{\scriptstyle Yf_j} & \\
& & YB & &
\end{array}
$$

The top square is the naturality square for β, and the bottom triangle results from applying Y to triangle condition for $a : A_i \to A_j$ with $f_i = f_j(Fa)$. So $y_i = y_j(Xa)$, and we have our definition of $\overline{\beta}$. Let's write $\overline{C}_B$ for the cocone we've just constructed.

Let us note the defining property of $\overline{\beta}$:

$$
\begin{aligned}
&\overline{\beta}_B l_i = y_i = (Yf_i)\beta_{A_i} \qquad \forall i \in I \\
&\overline{\beta}_B l_i : XA_i \to YB
\end{aligned}
$$

(a6) Given $b : B \to B'$, we need to verify commutativity in the naturality square

$$
\begin{CD}
LB @>{Lb}>> LB' \\
@V{\overline{\beta}_B}VV @VV{\overline{\beta}_{B'}}V \\
YB @>{Yb}>> YB'
\end{CD}
$$

To obtain Lb, we mapped D_B into a subdiagram of $D_{B'}$, which determined a cocone on D_B with vertex LB'. To obtain $\overline{\beta}_B$, we constructed a different cocone on D_B, with legs y_i and vertex YB. We next construct a cocone C on D_B with vertex YB', such that both $(Yb)\overline{\beta}_B$ and $\overline{\beta}_{B'}(Lb)$ are completing morphisms from the vertex LB (of C_B) and the vertex YB (of C). It follows that $(Yb)\overline{\beta}_B = \overline{\beta}_{B'}(Lb)$.

The cocone C is easiest to picture when $I \to I'$ is injective, since then C is a sub-cocone of $\overline{C}_{B'}$. In this case, each A_i belongs to $D_{B'}$, each $f'_i = bf_i$ and each $l'_i = (Lb)l_i$, and if we set $y'_i = (Yf'_i)\beta_{A_i}$, then y'_i is the leg of $\overline{C}_{B'}$ from the node A_i to the vertex YB'. So the nodes of C are $(A_i)_{i\in I}$, with legs $(y'_i)_{i\in I}$ from the A_i to YB'.

All the triangle conditions hold for C because they hold in $\overline{C}_{B'}$. Also the defining equations for $\overline{\beta}_{B'}$

$$\overline{\beta}_{B'} l'_{i'} = y'_{i'} \qquad \forall i' \in I'$$

include these equations

$$\overline{\beta}_{B'} l'_i = y'_i \qquad \forall i \in I$$

since $I \subseteq I'$. Noting that $l'_i = (Lb)l_i$, we have

$$\overline{\beta}_{B'}(Lb)l_i = y'_i \qquad \forall i \in I$$

and we see that $\overline{\beta}_{B'}(Lb)$ is a completing morphism from the vertex LB of C_B to the vertex YB of C.

On the other hand

$$y'_i = (Yf'_i)\beta_{A_i} = Y(bf_i)\beta_{A_i} = (Yb)(Yf_i)\beta_{A_i} = (Yb)y_i$$

so appending the morphism $YB \xrightarrow{Yb} YB'$ to the bottom of $\overline{C}_B$ also gives us C. Thus the defining equations for $\overline{\beta}_B$ yield

$$(Yb)\overline{\beta}_B l_i = (Yb)y_i = y'_i \qquad \forall i \in I$$

and $(Yb)\overline{\beta}_B$ is also a completing morphism. Therefore

$$(Yb)\overline{\beta}_B = \overline{\beta}_{B'}(Lb)$$

as required.

What changes if $I \to I'$ is not injective? Distinct f_i's can map to the same f_i', and so two nodes labeled A_i in D_B can map to the same node in $D_{B'}$. However, $I \to I'$ still induces a map from D_B to $D_{B'}$, which is the "top row" of $\overline{C}_{B'}$. This is all we need to construct C. The rest of the argument is not affected at all.

(a7) First we look at $(\overline{\beta}F)\eta \stackrel{?}{=} \beta$. To prove this, we have to show it holds at each $A \in \mathbf{A}$. Using the definitions of horizontal and vertical composition (see Leinster p.30 and p.37), this becomes

$$\overline{\beta}_{FA}\eta_A \stackrel{?}{=} \beta_A$$

We consider the cocone $\overline{C}_{FA}$, which determines $\overline{\beta}_{FA}$. It has vertex YFA and legs

$$y_i : \quad XA_i \xrightarrow{\beta_{A_i}} YFA_i \xrightarrow{Yf_i} YFA$$

where the family $(A_i)_{i\in I}$ comes from the family of all $(f_i : FA_i \to FA)_{i\in I}$. We earlier specified that f_1 is the identity 1_{FA}, so $A_1 = A$, and $Yf_1 = Y1_{FA} = 1_{YFA}$. Therefore the leg y_1 is just β_A.

We now appeal to the defining property of $\overline{\beta}$, $\overline{\beta}_{FA}l_i = y_i$. For the special index $i = 1$, we have $\eta_A = l_1$ and $y_1 = \beta_A$, and the equation reads $\overline{\beta}_{FA}\eta_A = \beta_A$, as desired.

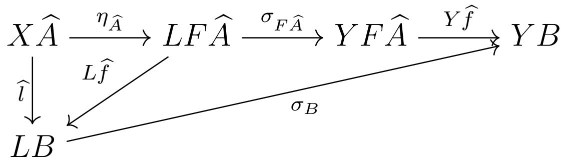
We turn our attention to $\overline{(\sigma F)\eta} \stackrel{?}{=} \sigma$. We must check this at each $B \in \mathbf{B}$. Let $\beta = (\sigma F)\eta$, so:

$$\eta : X \Rightarrow LF \qquad\qquad \sigma F : LF \Rightarrow YF$$
$$\sigma : L \Rightarrow Y \qquad\qquad \beta = (\sigma F)\eta : X \Rightarrow YF$$

We defined $\overline{\beta}_B$ via the cocones C_B and $\overline{C}_B$, the latter having legs $y_i = (Yf_i)\beta_{A_i}$ for each $f_i : FA_i \to B$. Since $\beta_A = \sigma_{FA}\eta_A$ for all A, we get

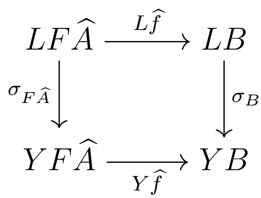
$$y_i : \quad XA_i \xrightarrow{\eta_{A_i}} LFA_i \xrightarrow{\sigma_{FA_i}} YFA_i \xrightarrow{Yf_i} YB$$

It will prove notationally convenient to let $\widehat{A}$, $\widehat{f}$, $\widehat{l}$, and $\widehat{y}$ stand for A_i, f_i, l_i, and y_i for some arbitrary $i \in I$. Now suppose we can show that this diagram commutes:



The top row is just the leg $\widehat{y}$ in $\overline{C}_B$, and the left vertical arrow is the leg $\widehat{l}$ in C_B. Thus σ_B would satisfy the defining condition for $[\overline{(\sigma F)\eta}]_B$. Consequently, our problem is reduced to proving commutativity of the diagram. We do this by showing that the two subtriangles commute.

By hypothesis $\sigma : L \Rightarrow Y$ is a natural transformation, so we have the naturality square



so $\sigma_B(L\widehat{f}) = (Y\widehat{f})\sigma_{F\widehat{A}}$. So the right subtriangle commutes.

Our definition of $L\widehat{f}$, for $\widehat{f} : F\widehat{A} \to B$, involves looking at the cocones $C_{F\widehat{A}}$ and C_B; we have a mapping from the first to the second induced by the mapping from $(F \Rightarrow F\widehat{A})$ to $(F \Rightarrow B)$. Let's denote by $(\widehat{f}_k)_{k \in K}$ the morphisms coming from $(F \Rightarrow F\widehat{A})$: an object of $(F \Rightarrow F\widehat{A})$ is

$$(\widehat{A}_k, \widehat{f}_k : F\widehat{A}_k \to B)$$

We let $\widehat{l}_k$ be the corresponding leg of $C_{F\widehat{A}}$, so $\widehat{l}_k : X\widehat{A}_k \to L\widehat{A}$.

The mapping from $(F \Rightarrow F\widehat{A})$ to $(F \Rightarrow B)$ is given by $\widehat{f}_k \mapsto \widehat{f}\widehat{f}_k$. Denote the corresponding leg mapping from $C_{F\widehat{A}}$ to C_B by $\widehat{l}_k \mapsto l_k$. (This was denoted $l_i \to l'_i$ in (a2).) So we have:

$$\begin{array}{lcl} \widehat{f}_k : F\widehat{A}_k \to F\widehat{A} & \mapsto & \widehat{f}\widehat{f}_k : F\widehat{A}_k \to B \\ \widehat{l}_k : X\widehat{A}_k \to L\widehat{A} & \mapsto & l_k : X\widehat{A}_k \to LB \end{array}$$

The l_k's are some of the legs of C_B, though perhaps with different multiplicities (i.e., the index set mapping $K \to I$ may not be injective). Recall also that the nodes $X\widehat{A}_k$ are some of the nodes of C_B, though again perhaps with different multiplicities.

The defining property of $L\widehat{f}$ is

$$l_k = (L\widehat{f})\widehat{l}_k$$

(In the notation of (a2), this was written $l'_i = (Lb)l_i$.)

The morphism $\widehat{f}$ has a double role. On the one hand, it's a part of $(F \Rightarrow B)$, since it goes from $F\widehat{A}$ to B. In this role, $\widehat{f}$ corresponds to $\widehat{l}$ of C_B. (Recall that $\widehat{f}$ and $\widehat{l}$ stand for f_i and l_i, for some $i \in I$.) On the other hand, $\widehat{f}$ determines the mapping of $C_{F\widehat{A}}$ into C_B.

For a particular index k (denote it by 1), we have $\widehat{A}_1 = \widehat{A}$ and $\widehat{f}_1 = 1_{F\widehat{A}}$. We then have

$$\begin{array}{lcl} \widehat{f}_1 = 1_{F\widehat{A}} & \mapsto & \widehat{f} \\ \widehat{l}_1 = \eta_{\widehat{A}} & \mapsto & l_1 = \widehat{l} \end{array}$$

The equation $\widehat{l}_1 = \eta_{\widehat{A}}$ follows from the definition of η in (a4). Here, l_1 is the leg corresponding to $\widehat{l}_1$ under the mapping $\widehat{l}_k \mapsto l_k$.

We now apply the defining property for the special index $k = 1$, and get:

$$\widehat{l} = (L\widehat{f})\eta_{\widehat{A}}$$

as required.

(b) Recall that L is an abbreviation for $\mathrm{Lan}_F X$. So (a) establishes a 1–1 correspondence

$$[\mathbf{B}, \mathcal{S}](\mathrm{Lan}_F X, Y) \cong [\mathbf{A}, \mathcal{S}](X, YF)$$

with the mappings $\sigma \mapsto (\sigma F)\eta$ going from left to right, and $\beta \mapsto \overline{\beta}$ from right to left. Our use of $\overline{\beta}$ is in keeping with Leinster's notation for adjoints (p.42); Leinster would also write $\overline{\sigma}$ for $(\sigma F)\eta$.

It's worth noting that η, as defined in part (a), is the unit of this adjunction, as we see from this diagram. (The top row belongs to $[\mathbf{B}, \mathcal{S}]$, the bottom to $[\mathbf{A}, \mathcal{S}]$.)

$$\begin{array}{ccc}
\mathrm{Lan}_F X & \xrightarrow{1_L} & \mathrm{Lan}_F X \\
{\scriptstyle \mathrm{Lan}_F}\uparrow & & \downarrow{\scriptstyle -\circ F} \\
X & \xrightarrow[\eta_X]{} & (\mathrm{Lan}_F X)F
\end{array}$$

It remains to check the naturality diagrams, (2.2) and (2.3), p.42.

(b1) Diagram (2.2):

$$\begin{array}{ccccc}
X & \xRightarrow{(\sigma F)\eta} & YF & \xRightarrow{\tau F} & Y'F \\
\downarrow & & \uparrow & & \uparrow \\
\mathrm{Lan}_F X & \xRightarrow[\sigma]{} & Y & \xRightarrow[\tau]{} & Y'
\end{array}$$

We have to verify this for each $A \in \mathbf{A}$. Noting that $\overline{\tau\sigma} = ((\tau\sigma)F)\eta$, we compute:

$$\begin{aligned}
[((\tau\sigma)F)\eta]_A &= ((\tau\sigma)F)_A\eta_A \\
&= (\tau\sigma)_{FA}\eta_A \\
&= \tau_{FA}\sigma_{FA}\eta_A \\
[(\tau F)(\sigma F)\eta]_A &= (\tau F)_A((\sigma F)\eta)_A \\
&= \tau_{FA}(\sigma F)_A\eta_A \\
&= \tau_{FA}\sigma_{FA}\eta_A
\end{aligned}$$

So (2.2) holds.

(b2) Diagram (2.3):

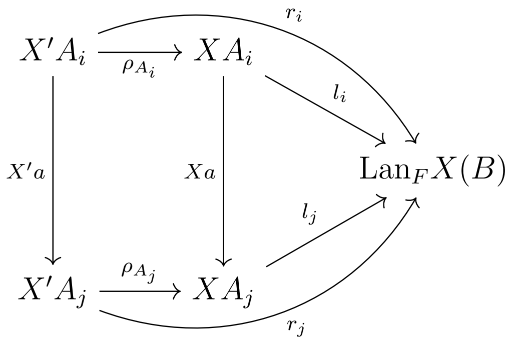
$$\begin{array}{ccccc}
X' & \overset{\rho}{\Longrightarrow} & X & \overset{\beta}{\Longrightarrow} & YF \\
\downarrow & & \downarrow & & \uparrow \\
\mathrm{Lan}_F X' & \underset{\mathrm{Lan}_F\rho}{\Longrightarrow} & \mathrm{Lan}_F X & \underset{\overline{\beta}}{\Longrightarrow} & Y
\end{array}$$

This must be checked for each $B \in \mathbf{B}$.

We see that we need to define $\mathrm{Lan}_F\rho$ for a natural transformation $\rho : X' \Rightarrow X$; $\mathrm{Lan}_F\rho$ will be a natural transformation $\mathrm{Lan}_F X' \Rightarrow \mathrm{Lan}_F X$. We start by picking an arbitrary $B \in \mathbf{B}$ and looking at $(F \Rightarrow B)$, i.e., at the family $(f_i : FA_i \to B)_{i\in I}$. Then $\mathrm{Lan}_F X'(B)$ is the colimit of $(X'A_i)_{i\in I}$, and $\mathrm{Lan}_F X(B)$ is the colimit of $(XA_i)_{i\in I}$, subject to the usual triangle conditions. Let's say the legs of the colimit cocones are respectively l_i' and l_i. Define r_i as the composition

$$r_i : X'A_i \xrightarrow{\rho_{A_i}} XA_i \xrightarrow{l_i} \mathrm{Lan}_F X(B)$$

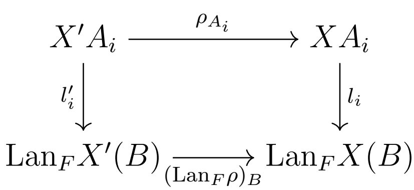
The following diagram shows that this is a cocone (where $a : A_i \to A_j$ is the morphism in a triangle condition of $(F \Rightarrow B)$, as usual):



So there exists a unique

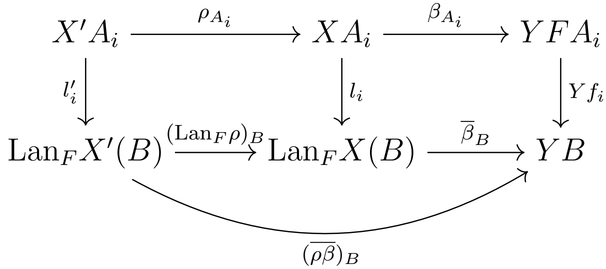
$$(\mathrm{Lan}_F \rho)_B : \mathrm{Lan}_F X'(B) \to \mathrm{Lan}_F X(B)$$

whose defining property is given by the commutativity of this diagram:



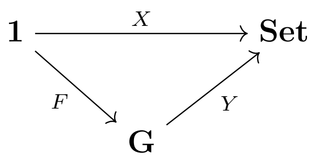
One must check that the $(\mathrm{Lan}_F \rho)_B$ combine to form a natural transformation, and that Lan_F is functorial, i.e., $\mathrm{Lan}_F(\rho\rho') = \mathrm{Lan}_F(\rho)\mathrm{Lan}_F(\rho')$. I leave these verifications as exericises.

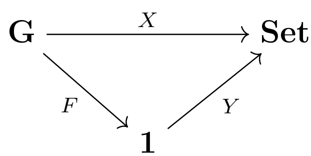
It remains to check (2.3). Consider this diagram:



The left rectangle commutes by the defining property of $(\mathrm{Lan}_F\rho)_B$; the right rectangle, by the defining property of $\overline{\beta}_B$. So $\overline{\beta}_B\,(\mathrm{Lan}_F\rho)_B$ satisfies the defining property of $(\overline{\rho\beta})_B$, i.e., the entire diagram commutes. Thus $(\overline{\rho\beta})_B = \overline{\beta}_B\,(\mathrm{Lan}_F\rho)_B$, and (2.3) is satisfied. So Lan_F is left adjoint to $Y \mapsto YF$.

(c) In the diagrams





1 is the category with a unique object $*$ and unique morphism 1_*, and **G** is G as a 1-object category (Example 1.1.8(c), p.14)—so the elements of G correspond to the morphisms of **G**. As we know from Examples 1.2.8 and 1.3.4 (pp.22 and 29), functors from **G** to **Set** are essentially G-sets, and natural transformations are essentially equivariant maps. Also a functor from **1** to **Set** is basically a set, namely the image of $*$. Let's write X and Y for these sets/G-sets—on the left X is the set and Y is the G-set, on the right vice versa.

On the left, the composition YF takes $*$ to Y and the morphism 1_* to 1_Y. So basically it takes $*$ to the G-set Y stripped of its G-action. In other words, $Y \mapsto YF$ amounts to the forgetful functor $Y \mapsto UY$ (from **G-set** to **Set**).

On the right, YF is a G-set with underlying set Y and the trivial action. In other words, $Y \mapsto YF$ amounts to the functor Triv from §2.5.

We described the left and right adjoints to U and to Triv in §2.5: $F \dashv U \dashv$ Map and Orb $\dashv$ Triv $\dashv$ Fix. So left and right Kan extensions give all these adjoints.

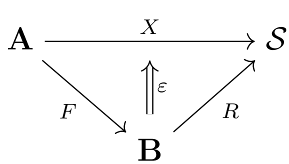
Incidentally, Leinster doesn't spell out what he means by the dual of part (a). We expressed the pivotal step as a 1–1 correspondence between the f_i's and the l_i's:

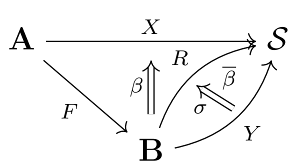
$$\begin{aligned} (F \Rightarrow B): \quad & (f_i : FA_i \to B)_{i \in I} \\ \text{Colimit cocone}: \quad & (l_i : XA_i \to LB)_{i \in I} \end{aligned}$$

To dualize, we keep the same basic functor triangle of F, X, and Y between $\mathbf{A}$, $\mathbf{B}$, and $\mathcal{S}$ (see below). However, the f_i's and the l_i's are now specified by

$$\begin{aligned} (B \Rightarrow F): \quad & (f_i : B \to FA_i)_{i \in I} \\ \text{Limit cone}: \quad & (l_i : RB \to XA_i)_{i \in I} \end{aligned}$$

Here $R = \mathrm{Ran}_F$ is the right Kan extension of F. Tracing through the argument of part (a), we end up with diagrams dual to those of part (a):





We replace η with ε since the unit has become a counit. Note that the natural transformations reverse direction, the functors don't. The argument of part (b) dualizes to show that Ran_F is right adjoint to $Y \mapsto YF$.

See also §2.5 and §6.2.

6.10 6.3.21, p.168

Functors with left adjoints preserve limits, and functors with right adjoints preserve colimits (Theorem 6.3.1, p.159).

(a) The trivial group 1 is a zero object of **Group**, that is, both initial and terminal. $U1$ is a singleton set, thus a terminal but not an initial object of **Set**. Since terminal objects are limits and initial objects are colimits, limits but not colimits are preserved for $1 \mapsto U1$. This accords with Theorem 6.3.1, provided that U has no right adjoint. (And note that U has a left adjoint.)

(b) To show that I from §3.4 (Exercise 3.2.16, p.78) has no right adjoint, we need a colimit in **Set** that is not preserved by I. As it happens, I preserves the initial object $\varnothing$. But turning to coproducts, i.e., disjoint unions, we have our example. $I(1+1) \neq I1 + I1$, since $I1 + I1$ is the discrete category on two objects and $I(1+1)$ has morphisms in both directions between its two objects.

To show that C has no left adjoint, we need a limit in **Cat** that is not preserved by C. We use monics (item 10 of §1.11). Let $\mathcal{A}$ be the discrete category on two objects, and let $\mathcal{B}$ be the indiscrete category on two objects. We have an obvious functor $F : \mathcal{A} \to \mathcal{B}$ that is the identity on objects and is an inclusion map on morphisms—$\mathcal{B}$ is "the same" as $\mathcal{A}$ except with two morphisms added. F is monic because the functor $\mathrm{Mor} : \mathbf{Cat} \to \mathbf{Set}$ that sends a category to its set of morphisms is faithful[12], and $\mathrm{Mor}(F)$ is injective

[12]Note that the functor $\mathrm{Ob} = O$ isn't.

(it's an inclusion map!), and faithful functors pull back monics (item 8 of §1.11). But $C\mathcal{A}$ has two elements and $C\mathcal{B}$ has one, so CF isn't monic.

Curiously, C preserves both terminal objects and products, the other two "go to" examples of limits. This is trivial for terminals, and makes a nice exercise for products.

(c) We recall (§2.7, Exercise 2.1.17, p.50) that X is a topological space and $\mathbf{Presheaf}(X)$ is the category of presheaves on X. Also, the functor Λ is defined thus: for any set A, $\Lambda A(\varnothing) = A$, $\Lambda A(Y) = \varnothing$ for each open set $Y \neq \varnothing$, and the morphisms in ΛA are all empty functions except for 1_A on $\Lambda A(\varnothing)$. The functor ∇ has $\nabla A(X) = A$ and $\nabla A(Y) = 1$ for each $Y \neq X$, where 1 is some arbitrary singleton; the morphisms in ∇A are all constantly 1 functions except for 1_A on $\nabla A(X)$. Finally, if $f : A \to B$, then the natural transformation Λf is defined by $(\Lambda f)_\varnothing = f$ and $(\Lambda f)_Y =$ the empty function for all $Y \neq \varnothing$. Likewise, $(\nabla f)_X = f$ and $(\nabla f)_Y = 1_1$ for all $Y \neq X$.

We exclude the case $X = \varnothing$. Λ does not preserve terminal objects, hence has no left adjoint. For if $A = 1$, then $\Lambda A(\varnothing) = 1$ and $\Lambda A(Y) = \varnothing$ for all $Y \neq \varnothing$, but the terminal presheaf on X clearly must have a singleton value everywhere, i.e., if $F \in \mathbf{Presheaf}(X)$ is terminal, $F(Y) = 1$ for all open $Y \subseteq X$. (The 1's don't have to be the same for different Y's, but since all singletons are isomorphic in **Set**, that doesn't matter. Note that this terminal F is actually $\Delta 1$. Δ has a left adjoint and so preserves terminals.)

Similarly, $\nabla\varnothing(X) = \varnothing$ and $\nabla\varnothing(Y) = 1$ for all $Y \neq X$, while the initial presheaf (namely $\Delta\varnothing$) has the value $\varnothing$ everywhere.

If you insist on not excluding $X = \varnothing$, then in that case $\Lambda = \Delta = \nabla$ and $\Pi = \Gamma$, so we have two functors that are left and right adjoints of each other, giving a two-way infinite chain.

6.11 6.3.22, p.168

(A)⇒(R): Suppose $F \dashv U$, so we have a bijection $\mathcal{A}(FX, A) \cong \mathbf{Set}(X, UA)$ for all $X \in \mathbf{Set}$ and all $A \in \mathcal{A}$. Let 1 be a singleton in **Set**. So

$$\mathcal{A}(F1, A) \cong \mathbf{Set}(1, UA) \cong UA$$

since, as Leinster often points out, we have a natural 1–1 correspondence between elements of a set and maps from a singleton into the set. The correspondence is natural in A because of the naturality property of adjoints.

Since $\mathcal{A}(F1, A) = H^{F1}(A)$, this gives an isomorphism between the representable functor H^{F1} and U.

Observe that we want H^{F1} and not H_{F1}, as U is covariant.

(R)⇒(L): This is Prop.6.2.2 (p.148).

(b) We want a left adjoint F to $U \cong H^A$, with $A \in \mathcal{A}$. For a snappier proof, we'll just let $U = H^A$.

Observer first that for any $B \in \mathcal{A}$, $\mathcal{A}(A, B) = H^A(B) = UB \cong \mathbf{Set}(1, UB)$, where 1 as usual is a singleton. This suggests setting $F1 = A$. An arbitrary X is the sum of its elements, and $\mathcal{A}$ has sums by hypothesis, so we let FX be the sum of copies of $F1 = A$, one copy for each $x \in X$.

We nail down the details. Make X into a discrete category. In $\mathcal{A}$, our diagram is $D_X : X \to \mathcal{A}$, with the image of each $x \in X$ being A. (So D_X is the diagonal usually denoted ΔA.) We choose a colimit cocone for D_X $\check{F}X$, and denote the vertex by FX and the legs by i_x:

$$\check{F}X = \left(x \xrightarrow{i_x} FX \right)_{x \in X}$$

So FX is the sum of $|X|$ copies of A. We do this for each set X. Adapting the proof of Prop.6.1.4 (p.145) (or §5.17, ex.5.3.8, p.140), there's a natural way to define Ff for functions $f : X \to Y$ in **Set**: construct a cocone on diagram D_X with vertex FY by letting the x-leg $l_x : A \to FY$ be $i_{f(x)}$. (That is, the leg from A to FY in $\check{F}Y$ associated with $y = f(x) \in Y$.[13]) Since $\check{F}X$ is a colimit, there is a unique $Ff : FX \to FY$ such that $(Ff)i_x = i_{f(x)}$ for all $x \in X$. This makes F into a functor.

We now have to define the 1–1 correspondence $\mathcal{A}(FX, B) \cong \mathbf{Set}(X, UB)$. Given $a : FX \to B$, for every $x \in X$ we define $a_x = ai_x : A \to B$. (Think of it as $a_x : A_x \to B$.) So $a_x \in H^A(B) = UB$, and $x \mapsto a_x$ is a function $\bar{a} : X \to UB$. In the reverse direction, given $f : X \to UB = H^A(B)$, since $f(x) : A \to B$, we have a cocone $(A \xrightarrow{f(x)} B)_{x \in X}$ with vertex B. Since $\check{F}X$ is a colimit, there is a unique $\bar{f} : FX \to B$ making things commute. The verifications that $\bar{\bar{a}} = a$ and $\bar{\bar{f}} = f$, and of the naturality of the correspondence, are all routine.

6.12 6.3.23, p.168

(a) Let $\mathcal{A}$ be a preordered set; regarded as a category, we have a unique morphism $A \to B$ exactly when $A \leq B$, and no other morphisms (Example 1.1.8(e), p.15). We let $\equiv$ be the equivalence relation $A \equiv B$ iff $A \leq B$ and $B \leq A$, and let $\mathcal{A}/\equiv$ be the category whose objects are the equivalence classes $[A]$ with $[A] \leq [B]$ iff $A \leq B$, obviously independent of the choice of representatives A and B. Now let F be the mapping $A \mapsto [A]$; for each $[A] \in \mathcal{A}/\equiv$ choose an $A \in [A]$, call it GA. It is easily verified that F and G are functors, and that $GF \cong 1_{\mathcal{A}}$, $FG \cong 1_{\mathcal{A}/\equiv}$. Lemma 1.3.11 (§1.3) makes this even easier: since A and GFA both belong to $[A]$, we have $GF \cong 1_{\mathcal{A}}$; $FG[A] = [A]$, so $FG \cong 1_{\mathcal{A}/\equiv}$.

[13]Personally, I like to picture the $|X|$ copies of A in D_X as distinct dots A_x, even though they're all the same A; likewise for the A_y's in D_Y. Then the legs i_x in $\check{F}X$ all have visually distinct sources, likewise for the i_y's in $\check{F}Y$. Finally, think of l_x as $i_{f(x)}1_A$, where 1_A goes from A_x to $A_{f(x)}$.

(b) Let $\mathfrak{i}$ be the cardinality of I. Recall that the diagram $\Delta B : I \to \mathcal{A}$ sets $(\Delta B)i = B$ for all $i \in I$. There are $2^{\mathfrak{i}}$ different cones on ΔB with vertex A, since in $(A \xrightarrow{l_i} B)_{i \in I}$, we can choose $l_i = f$ or $l_i = g$ independently for each l_i. Let $a : A \to B^I$ be the completing morphism, so $l_i = p_i a$ for each i. (Of course, $p_i : B^I \to B$ is the projection to the i-th factor.) So each cone must have a different a, and there are at least $2^{\mathfrak{i}}$ different morphisms $A \to B^I$. Therefore $\mathcal{A}$ cannot have a set's worth of morphisms, i.e., is not small.

(c) From (b) we conclude that the category must be a preorder. From (a) we conclude that it is equivalent to a poset. Having all small products is equivalent, for a poset, to having all small meets. I.e., the poset is complete.

(d) All the arguments of (a)–(c) go through with the sole change of replacing "set's worth" and "small" with "finite".

6.13 6.3.24, p.169

(a) Every element of subgroup generated by $\{g_a | a \in A\}$ can be written as a product of g_a's and g_a^{-1}'s. If we say a *letter* is a g_a or a g_a^{-1} (see §2.2), then the cardinality of the generated subgroup is at most that of the set of all finite strings of letters. This cardinality is obviously at most $\max(\aleph_0, |A|)$.

(b) Suppose G is a group with $|G| \le |S|$. So there is an injection $f : G \to S$, and we can use f to transfer the group structure of G to $f(G)$. That is, there is a group G' isomorphic to G whose underlying set is a subset of S.

Now consider the collection of all groups whose underlying sets are subsets of S. Such a group is specified completely by $S' \subseteq S$ and its multiplication table, which in turn is the set $\{(x, y, z) | xy = z\}$ and hence a subset of $S' \times S' \times S'$. So starting with S, we form the power set, then for each element S' of the power set we form the power set of $S' \times S' \times S'$, and finally we consider all possible pairs (S', T) where $S' \subseteq S$ and T is a multiplication table on S'. By basic set theory, all these are sets.

(c) The comma category $(A \Rightarrow U)$ consists of all pairs $(G, f : A \to UG)$, where G is a group and f is a function from A to UG. In other words, elements of $(A \Rightarrow U)$ are groups G with the set A "dropped into" G. Something like a pointed set, except that instead of distinguishing just one element, we've "labeled" an element of G for every $a \in A$. I'll call f a *labeling*, and $(G, f : A \to UG)$ a *labeled group*. (§2.15 introduced this terminology.)

For any labeled group, the subgroup generated by the labeled elements (call it G') has cardinality at most $\max(\aleph_0, |A|)$ (by (a)). Let S be a set of cardinality $\max(\aleph_0, |A|)$; it follows that for any labeled group G, there is a group H with $UH \subseteq S$ and with H isomorphic to the generated subgroup G'. So there is a monomorphism $H \to G$; moreover, we can obviously transfer the labeling of G over to H, i.e., there is an $h : A \to UH$ making the obvious diagram commute. So $(H, h : A \to UH)$ belongs to $(A \Rightarrow U)$, and there is a morphism in $(A \Rightarrow U)$ from the labeled group (H, h) to the labeled group (G, f).

Slightly extending part (b), we see that the collection of all labeled groups (H, h) with $UH \subseteq S$ is a set; we have a set's worth of H's with $UH \subseteq S$, and for each such H, a set's worth of labelings $h : A \to UH$. So the collection of all such (H, h) is a weakly initial set for $(A \Rightarrow U)$.

(d) All the GAFT hypotheses hold: **Group** is locally small, small complete, and the forgetful functor U : **Group** $\to$ **Set** preserves limits[14]. So U has a left adjoint, the free group functor.

(e) Let's compare this construction of the free group functor with the second one given in §2.2, the "subgroup of a direct product"; I'll call this the Tits construction, after a remark in Lang [5, p.66] that suggests he invented it. Several points of comparison leap out. The Tits construction begins with

[14]The last two facts fall out from §§5.20&5.21 (Exercises 5.3.11 and 5.3.12, p.140; 5.3.12 is the same as Lemma 5.3.6, p.139). **Set** has all small limits and U creates limits, so **Group** has all small limits and U preserves them.

a set S of cardinality $\max(\aleph_0, |A|)$, then looks at the set $\{G_i | i \in I\}$ of all groups whose underlying sets are subsets of S, and then for each i looks at all possible mappings $f : A \to UG_i$, making a copy G_{if} of G_i for each such f. So the set $\{G_{if}\}$ is really just the weakly initial set we constructed in the category $(A \Rightarrow U)$.

The next step in the Tits construction: form the product $G = \prod_{i,f} G_{if}$. The free group FA is not G, but the subgroup H generated by $\eta(A)$, where $\eta : A \to UG$ is defined by $\eta(a)_{if} = f(a)$. So it's a restricted subgroup of G. Let's see how the product enters in the proof of GAFT.

Lemma A.1 (p.172) sits at the heart of GAFT: a complete locally small category with a weakly initial set has an initial object. The proof of GAFT applies this to the category $(A \Rightarrow U)$. A moment's thought reveals that an initial object of $(A \Rightarrow U)$ is a free group: $(FA, \eta_A : A \to UFA)$ is initial iff FA satisfies the desired universal property.

The initial object is constructed as a limit cone (eq.A.1, p.172, adjusting notation to dovetail with our discussion):

$$\left(FA \xrightarrow{p_{if}} G_{if}\right)_{i,f}$$

Now, limits in Group are computed via eq.5.16 (p.121, Examples 5.1.22 and 5.1.23) as an equalizer of a product. In this case:

$$FA = \{(x_{if})_{i,f} | x_{if} \in G_{if} \, \forall i, f \text{ and } u(x_{if}) = x_{jg} \, \forall G_{if} \xrightarrow{u} G_{jg}\} \tag{5}$$

In other words, the limit is the subgroup of the product $G = \prod_{i,f} G_{if}$ subject to a bunch of a constraints, namely

$$u(x_{if}) = x_{jg} \text{ whenever } u : G_{if} \to G_{jg} \tag{6}$$

We've now reached the point where the Tits and GAFT constructions both serve up subgroups of $\prod_{i,f} G_{if}$; we need to compare the constraints defining the subgroups. First, GAFT. The u's in eq.6 are all the homomorphisms that respect labels: $u(f(a)) = g(a)$ for all $a \in A$. (That's part of the definition of morphisms in $(A \Rightarrow U)$.) Next, Tits. The subgroup H is generated by $\eta(A)$, where $\eta : A \to UG$ is defined by $\eta(a)_{if} = f(a)$. Observe that each element of $\eta(A)$ satisfies all the constraints of eq.6, because the u's respect labels, and $\eta(a)_{if} = f(a)$, $\eta(a)_{jg} = g(a)$. The same is thus true of the subgroup generated by $\eta(A)$, because the u's are homomorphisms. So H is a subgroup of the GAFT group FA of eq.5. (In fact it's equal to it—we'll see that soon.) So the Tits construction enforces all the constraints of eq.6, though indirectly.

Let's say that G_{if} *dominates* G_{jg} if there is a label-respecting homomorphism $u : G_{if} \to G_{jg}$. If G_{if} dominates G_{jg}, then we can throw out the jg component of tuple $(x_{kh})_{k,h}$—it's completely determined by the if component! In other words, the projection of FA to the product of the undominated G_{if}'s is an isomorphism.

Once we know that there is a free group with the required universal property, we are ready for the punchline: there is exactly one undominated G_{if}! Suppose F is free over A with labeling $e : A \to F$. It's easy to see that the subgroup generated by $e(A)$ also has the universal property, so $|F| \leq \max(\aleph_0, |A|)$. So there's a copy of F among the G_{if}'s, and it dominates every other G_{if} by the universal property. It follows readily that the Tits group H is identical to the GAFT group FA.

§A.4 opines on this and another application of GAFT.

A Proof of GAFT

A.1 A.3, p.173

(a) First note that $(0 \xrightarrow{u_B} B)_{B \in \mathcal{B}}$ is a cone on the identity diagram, where u_B is the unique morphism from 0 to B. The commutativity conditions hold because u_B is unique. Observe that $u_0 = 1_0$.

Next, let $(V \xrightarrow{l_B} B)_{B \in \mathcal{B}}$ be a cone on the identity diagram, so for any morphism $f : A \to B$ we have $f l_A = l_B$. In particular, setting $A = 0$ (so $f = u_B$) we have $u_B l_0 = l_B$. For $(0 \xrightarrow{u_B} B)_{B \in \mathcal{B}}$ to be a limit cone, we need a unique $g : V \to 0$ with $u_B g = l_B$ for all B. We've just seen that $u_B l_0 = l_B$ for all B, so we can set $g = l_0$; this is the only possible choice, as we see by setting $B = 0$ in $u_B g = l_B$ and remembering that $u_0 = 1_0$.

(b) We have $f p_A = p_B$ for any $f : A \to B$, so setting $A = L$ and $f = p_B$, we have $p_B p_L = p_B$ for all B. Now we apply the universal property of limit cones, with both cones being $(L \xrightarrow{p_B} B)_{B \in \mathcal{B}}$. Both 1_L and p_L can serve as the unique mediating morphism g, i.e., the g that satisfies $p_B g = p_B$ for all B. So $p_L = 1_L$.

Now pick an arbitrary B, and look at the set of all $f : L \to B$. These all satisfy $f p_L = p_B$, or since $p_L = 1_L$, $f = p_B$. In other words, for all B there is a unique morphism from L to B; thus L is an initial object.

A.2 A.4, p.173

(a) $(\forall x \in C)(\exists s \in S) s \leq x$.

(b) Let $m = \bigwedge_{s \in S} s$. By the definition of meet, $(\forall s \in S) m \leq s$, so for any $x \in C$, we have $m \leq s \leq x$ for some $s \in S$. So $(\forall x \in C) m \leq x$, thus m is the least element of C.

A.3 A.5, p.173

(a) The picture below illustrates the case where $\mathbf{I}$ is the category $1 \to 2$, i.e., we have two objects and a unique morphism from 1 to 2. The diagram $E : \mathbf{I} \to \mathcal{B}$ then has two objects with a morphism, $B_1 \xrightarrow{f} B_2$, and $G \circ E$ has $GB_1 \xrightarrow{Gf} GB_2$:

$$
\begin{array}{ccccc}
 & & G \circ E & & E \\
 & & & & \\
A & \xrightarrow{a_1} & GB_1 & & B_1 \\
 & \searrow^{a_2} & \downarrow Gf & & \downarrow f \\
 & & GB_2 & & B_2
\end{array}
$$

The objects in $(A \Rightarrow G)$ are the pairs (B_1, a_1) and (B_2, a_2), and the definition of morphism in $(A \Rightarrow G)$ says that $(Gf)a_1 = a_2$, so we have a cone on $G \circ E$.

The general case is no different, just lots of these pictures glued together.

(b) To avoid drawing a messy and confusing picture, I will impose upon the reader's powers of visualization. Imagine that the B_2 in the previous picture is replaced with a swarm of B_2's (i.e., a diagram in $\mathcal{B}$), and that B_1 (with attendent f's) is the limit cone.

The (strict) definition of "creates limits" starts with a diagram in $(A \Rightarrow G)$, in other words, just with the diagram of B_2's and the a_2's from A to the GB_2's, forming a cone in $\mathcal{A}$ with vertex A. Applying the projection P_A gives us the diagram of B_2's in $\mathcal{B}$. As said, we now assume that B_1 is the vertex of a limit cone E in $\mathcal{B}$ with the legs being the f's. Apply the limit-preserving functor G gives us a limit cone $G \circ E$ in $\mathcal{A}$, with limit GB_1.

Because $G \circ E$ is a limit cone with vertex GB_1, and the a_2's are the legs of a cone on the same base with vertex A, there is a unique $a_1 : A \to GB_1$

making everything commute. The whole commuting shebang—vertex A, morphism a_1, GB_1, all the GB_2's, all the a_2's and all the Gf's—is a limit cone in $(A \Rightarrow G)$ with vertex (B_1, a_1). It's unique because a_1 is uniquely determined once we know B_1, and B_1 is determined by the choice of limit cone in $\mathcal{B}$. So we've lifted the limit cone in $\mathcal{B}$ uniquely to a limit cone in $(A \Rightarrow G)$.

A.4 Another Application of GAFT: Hausdorffification

In Example 6.3.11, Leinster sketches the usual free group construction, and remarks, "But using GAFT, we can avoid these complications entirely." Here he echoes Mac Lane's comment [4, p.123], "[O]ur theorem has produced this free group without entering into the usual (rather fussy) explicit construction of the elements of FX as equivalence classes of words in letters of X." I invite the reader to look at the standard construction as sketched at the beginning of §2.2. Compare this with *all the ingredients* to the GAFT proof: the proof of GAFT itself, the verification that the comma categories have weakly initial sets, and the verifications that U creates limits and that **Set** has them. Now say which is less complicated, more understandable, and better motivated.

At least Leinster hasn't drunk *all* the Kool Aid. He also says, "The price to be paid is that GAFT does not give us an explicit description of free groups".

So thumbs down on *this application* as a justification for GAFT.

Am I saying the GAFT isn't cool? On the contrary—the GAFT is very cool. And the notion of adjoint functor is *way* cool. One is almost tempted to resuscitate 60s slang to say how cool it is.

What makes the idea of adjoint functor so groovy is the way it sweeps up such disparate constructs into a single bag: free groups, universal enveloping algebras, Stone-Čech compactifications, and so much else. That one existence theorem covers all these situations—that's the ice cream on the pie. But in any concrete instance, you're likely to find a tailored existence proof more understandable and glean more insight from it.

Here's another illustration: Hausdorffification. Let $U : \mathbf{Hausdorff} \to \mathbf{Top}$ be the forgetful functor. We saw at the end of §5.21 that **Hausdorff** is complete and that this U preserves limits. To show that U has a left adjoint H (the **Hausdorffification** functor) we have to verify the remaining hypothesis of GAFT. The category $(A \Rightarrow U)$ consists of pairs $(X, f : A \to UX)$, in other words, maps from a fixed topological space A to an arbitrary Hausdorff space X. We can always factor $f : A \to UX$ as $A \to f(A) \to UX$, and we can replace $f(A)$ with any space homeomorphic to it. Since the cardinality of $f(A)$ is at most $|A|$, we just have to pick a fixed set S of cardinality $|A|$ and consider all possible Hausdorff topologies on all possible subsets of S. This is clearly a set. So $(A \Rightarrow U)$ possesses a weakly initial set, and GAFT applies.

Here are two other constructions, both more concrete. Let X be a topological space, and say $x \sim y$ if x and y cannot be separated by disjoint open sets. Since $\sim$ might not be transitive, let $\approx$ be the transitive closure of $\sim$. Let $h : X \to h(X)$ be the quotient map. (Let's call this "the h construction".)

Now, $h(X)$ possesses the desired universal property. Proof: suppose f maps X to a Hausdorff space Y. We will show that $x \approx y$ implies $f(x) = f(y)$. This will do the trick, because it means that f maps each equivalence class to a single element of Y; this makes it obvious that there is a unique function $\overline{f} : h(X) \to Y$ such that $f = \overline{f}h$. Furthermore $\overline{f}$ is continuous because $h(X)$ has the quotient topology. The implication $x \approx y \Rightarrow f(x) = f(y)$ follows from the same implication for $\sim$, since $\approx$ is the *smallest* equivalence relation containing $\sim$, and $f(x) = f(y)$ is an equivalence relation between x and y. We prove this last implication in contrapositive form. Suppose $f(x) \neq f(y)$. Since Y is Hausdorff, there are disjoint open sets U and V around $f(x)$ and

$f(y)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets around x and y. So $x \not\sim y$.

Hence if $h(X)$ is Hausdorff, then it's the Hausdorffification of X. But $h(X)$ might not be Hausdorff! Example: start with any Hausdorff space S (e.g., the real line) and any sequence s_n in S converging to an element s. Let X have S as its underlying set but with a coarser topology. Sets are open in X iff they are open in S and they include either all the s_n's or none of them. So $s_i \sim s_j$ for all i and j, but $s_i \not\sim x$ for any x that is not an s_j. The $\approx$-equivalence classes are easy to describe: all the s_i's are in one class and every other x is in a singleton class by itself (since S was Hausdorff). So all the s_i's map to a single element of $h(X)$ and s, in particular, maps to a different element. Because $s_n \to s$, these two elements cannot be separated by disjoint open sets.

We proceed via transfinite induction, defining an ordinal sequence of equivalence relations $\approx_\alpha$ on X and quotient maps $h_\alpha : X \to X_\alpha$. We start off with $\approx_0$ being equality. We define $\approx_{\alpha+1}$ in terms of $\approx_\alpha$: $x \approx_{\alpha+1} y$ iff $h_\alpha(x) \approx h_\alpha(x)$, where $\approx$ is the result of applying the h construction to X_α. For a limit ordinal λ, define $x \approx_\lambda y$ iff $x \approx_\alpha y$ for some $\alpha < \lambda$ (with $x, y \in X$). Note that each each equivalence relation is a coarsening of all the previous ones: $x \approx_\alpha y$ implies $x \approx_\beta y$ for any $\alpha < \beta$.

All the $\approx_\alpha$ relations enjoy the same property as $\approx$: if $x \approx_\alpha y$, then $f(x) = f(y)$ for any f mapping X to a Hausdorff space. This follows by transfinite induction. If $x \approx_{\alpha+1} y$ then $h_\alpha(x) \approx h_\alpha(y)$ (definition) which implies $\hat{f}(h_\alpha(x)) = \hat{f}(h_\alpha(y))$, where $\hat{f}$ is the unique map such that $f = \hat{f} h_\alpha$. So $f(x) = \hat{f}(h_\alpha(x)) = \hat{f}(h_\alpha(y)) = f(y)$. For a limit ordinal, if $x \approx_\lambda y$, then $x \approx_\alpha y$ for some $\alpha < \lambda$, so $f(x) = f(y)$.

There exists a σ such that $\approx_\sigma$ and $\approx_{\sigma+1}$ are the same. Proof: the number of different equivalence relations on X is at most $2^{|X|^2}$ (each equivalence relation is a set of ordered pairs). So there must be a $\sigma < \tau$ with $\approx_\sigma$ and $\approx_\tau$ the same, and hence (since each $\approx_\alpha$ is a coarsening of all the previous equivalence relations), $\approx_\sigma$ equals $\approx_{\sigma+1}$. So $h_\sigma(X)$ is Hausdorff, and by what we've already shown, it has the required universal property.

Now for the second construction. Define $x \equiv y$ iff $x \approx y$ for every equivalence relation $\approx$ such that $X/\approx$ is Hausdorff. First we prove the universal property. Suppose $x \equiv y$ and f maps X to a Hausdorff space Y. The relation $f(x) = f(y)$ is an equivalence relation on X, call it $\approx_f$, and $X/\approx_f$ has an obvious injection into Y, continuous because $X/\approx_f$ has the quotient topology. So $X/\approx_f$ is Hausdorff. So $x \equiv y$ implies $x \approx_f y$, i.e., $f(x) = f(y)$.

Next we show that $X/\equiv$ is Hausdorff. Suppose $x \not\equiv y$. By definition then, $x \not\approx y$ for some $\approx$ with $X/\approx$ Hausdorff. Note that each $\approx$-equivalence class is a union of $\equiv$-equivalence classes. This gives us a factorization of the quotient map $q : X \to X/\approx$ into $X \xrightarrow{e} X/\equiv \xrightarrow{p} X/\approx$, e and p being quotient maps. Since $x \not\approx y$, we have $q(x) \neq q(y)$, so there are disjoint open sets U and V around $q(x)$ and $q(y)$. It follows that $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint open sets around $e(p)$ and $e(y)$. But $e(p)$ and $e(y)$ are arbitrary distinct elements of $X/\equiv$. QED

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