Enumeration Without Duplication

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This is my retelling of the last part of this paper:

Richard Friedberg, “Three Theorems on Recursive Enumeration: I, Decomposition; II, Maximal Sets; III, Enumeration Without Duplication”, *Journal of Symbolic Logic,* vol. 23 (1958), pp. 309–316.

Theorem: There is a partial recursive $\psi(x,y)$ such that if $S_x = \{y|\psi(x,y) \text{ converges}\}$, then

(i) $\forall i\ \exists x \quad W_i = S_x$

(ii) $x \neq y \Rightarrow S_x \neq S_y$

That is, the r.e. sets can be uniformly enumerated without duplication.

Proof: We want

1. $\forall i\ \exists x \quad W_i = S_x$
2. $x \neq y \Rightarrow S_x$ and S_y are not the same finite set
3. $x \neq y \Rightarrow S_x$ and S_y are not the same infinite set

We will simultaneously enumerate all the S_x's in the usual dovetailing way; i.e., at step n we focus attention on a particular S_{x_n}, *activating* x_n if S_{x_n} has not been considered before (x_n thereafter remains *active*); each x is focussed on infinitely many times, and only a finite number of x's are active at any time. S_x^n is the subset of S_x enumerated by the end of step n, so for all n, almost all S_x^n are empty. We never remove elements from S_x, so $S_x = \bigcup_n S_x^n$. Similarly, let W_i^n be the subset of W_i enumerated in n steps.

We give first a simplified procedure which yields (1) and (2) but not (3). We impose

4. x and y both active at step $n \Rightarrow S_x^n \neq S_y^n$

(4) implies (2). At step n, we will focus attention on a particular W_i, dovetailing as above. At any time, some of the active W_i's will have *followers* S_x; we say x *follows* i. At any one time a given i has at most one follower, but i may lose a follower x (which thereafter is *free*, and never follows any i again) and may later gain another follower. If x follows i forever, we say x is *loyal*, else *disloyal*. When i is assigned a follower x, x is activated—this is the only way x's become active, so the active x's are just those that are or were followers. The idea is, if x follows i then we will try to make $S_x = W_i$. In any case, we will have always

5. If x follows i at the end of step n, then $S_x^n \subseteq W_i^n$

Suppose i is focussed on at step n. We then wish to: (i) give i a follower x (specifically, the smallest inactive x) if i doesn't already have one; (ii) set $S_x^n = W_i^n$. We have a conflict with (4) if there is an active y that does not follow i and $S_y^{n-1} = W_i^n$. Let y follow j if y is not free. We can resolve the conflict in two ways:

(a) Do nothing. That is, don't assign i a follower, and if x already follows i, don't update S_x^{n-1} (i.e., let $S_x^{n-1} = S_x^n$). We would say here that y *frustrates* i, and if y follows j, also that j *frustrates* i (through y).

(b) Perform (i) and (ii), and add something to each S_y such that $S_y^{n-1} = W_i^n$ so as to preserve (4). If such a y follows j, release y from following, i.e., make y free. We say here that i *injures* such y, and if y follows j, also that i *injures* j.

If we make choice (b), we say that i *succeeds* at step n, else i *fails* at step n. We say i is *successful* if i succeeds infinitely often. If i is successful and has a loyal follower x, then $W_i = S_x$. This cannot possibly occur for all i, for (4) holds and hence (2) holds also, and so if $W_i = W_j$ is finite, $i \neq j$, then we cannot have $S_x = W_i = W_j = S_y$, x following i, y following j and so $x \neq y$.

We resolve this by giving i higher priority than j if $i < j$, so: frustrate i only if i, to succeed, would have to injure $j < i$; otherwise do (b), injuring any $j > i$ as needed. This specifies the simplified construction completely.

To summarize: at step n, we focus on i; we compute W_i^n; we see if i would have to injure any $j < i$ to succeed; if so, we frustrate i; if not, we set $S_x^n = W_i^n$, activating the smallest inactive x to be i's follower unless i already has a follower x, and perform any injuries necessary to maintain (4); this last action will lose j a follower if j gets injured, and increase any S_y^{n-1}'s where y gets injured. Note right off that: (i) if i injures j, then there exists an m less than n such that $W_i^n = W_j^m$ (for if y follows j when i injures j, then $W_i^n = S_y^{n-1} = W_j^m$ for some $m < n$); (ii) if j gets injured infinitely often, then j must be successful and have an infinite number of followers; (iii) (4) and (5) hold.

6. If i injures j infinitely often, then $W_i = W_j$.

7. If j frustrates i infinitely often and j is successful, then $W_j = W_i$.

For both of these imply that for all N there are m, n greater than N such that $W_i^n = W_j^m$. (Note that j being successful guarantees that $(\forall N)(\exists m > N)(\exists y)$ (y follows j at the end of step m and so $S_y^m = W_j^m$).)

Now to prove (1): suppose i is a minimal index for W_i, i.e., $j < i$ implies $W_j \neq W_i$. Then i cannot be injured infinitely often (by (6)), so if i is successful, i has a loyal follower x and $W_i = S_x$. If i is unsuccessful, then some $j < i$ frustrates i infinitely often. So j must be unsuccessful, so j has only finitely many followers. So one of these followers, say y, frustrates i infinitely often. So $S_y = W_i$, clearly.

From (1) we see that all x's get activated eventually (else almost all S_x's would be empty), so (4) implies (2). This completes the proof that this construction gives (1) and (2). It remains to modify the construction to get (3) to hold also.

Our modifications will preserve (4)–(7). Also, we will impose:

8. If y ever becomes free, then S_y is finite.
9. If $W_i = W_j$, then i and j do not both get loyal followers.

Up till now, a free y could be injured at will. We change this, allowing i to injure y only if $y \geq i$, and even then at most once. This guarantees (8), but introduces more frustration: y can frustrate i if: (i) y follows $j < i$, or (ii) y is free and $y < i$, or (iii) i has previously injured y. The proof of (1) survives these changes, for the key point was that if i is a minimal index for W_i and i is unsuccessful, then only a finite number of y's can frustrate i. Since i is unsuccessful, i injures only a finite number of y's; if y follows j, then j must be unsuccessful by (7), so j has only a finite number of followers. Thus the y's given by (i) and (iii) form a finite collection.

To get (9), we introduce our last modification: free y if y follows j at step n, $j > i$, and $W_j^n \cap [0, y] = W_i^n \cap [0, y]$, where $[0, y] = \{t | 0 \leq t \leq y\}$. We say also that i injures j. If $W_i = W_j$, $i < j$, then this prevents j from having a loyal follower. Now that we have a new mode of injury, we must reprove (6). Suppose i injures j infinitely often; then i injures j infinitely often either in the old mode or the new. Suppose i injures j infinitely often in the new mode. So the followers of j tend to infinity, and so $(\forall N)(\exists n, y > N)\ W_i^n \cap [0, y] = W_j^n \cap [0, y]$; therefore $W_i = W_j$. ∎

Note that the mapping $i \mapsto x$ defined by $W_i = S_x$ cannot be recursive, since "$W_i = W_j$" is a complete Π_2^0 predicate.

The proof may be modified to yield: there is a partial recursive $\psi(x, y)$ such that

$$\forall i\ \exists x\ \forall y \quad \phi_i(y) \equiv \psi(x, y)$$

where '$\equiv$' means that one computation converges if and only if the other does, in which case the two sides are equal.