Stirling's Formula: Ahlfors' Derivation

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I found Ahlfors' derivation of Stirling's formula a little too elliptical in spots; also there were a couple of misprints in the second edition (§2.5, pp. 199–204). These notes fill in the details and make some supplementary remarks.

1 Preliminary Remarks

Gamma and Factorial: Recall that $n! = \Gamma(n+1)$, so these are all equivalent:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad \log n! = n\log n - n + \frac{1}{2}\log n + \frac{1}{2}\log 2\pi + o(1)$$

$$\Gamma(n) \sim \sqrt{2\pi}\frac{1}{\sqrt{n}}\left(\frac{n}{e}\right)^n \qquad \log\Gamma(n) = n\log n - n - \frac{1}{2}\log n + \frac{1}{2}\log 2\pi + o(1)$$

Complex Arguments: The other notes on Stirling's formula derived $n! \approx \sqrt{n}\left(\frac{n}{e}\right)^n$ by elementary arguments. Ahlfors' result is sharper in three ways:

- $\sqrt{2\pi}$ is derived.

- The result is proved for suitable complex z, not just for positive integer n.
- The limit implied by the $\sim$ sign, or by the $o(1)$, is proved (when suitably interpreted).

To elaborate on the last two points: let's write $J(z)$ for the "remainder term":

$$J(z) = \log\Gamma(z) - z\log z + z + \frac{1}{2}\log z - \frac{1}{2}\log 2\pi$$

The $o(1)$ means that $\lim_{n\to\infty} J(n) = 0$. Here n tends to infinity through positive integer values. Now, it is *not* correct simply to replace n by z; if z wanders off to infinity, $J(z)$ will not necessarily tend to 0. This is obvious if we reflect that $\Gamma(z)$ has poles at all non-positive integers, but Stirling's approximation has a singularity only at 0.

The correct generalization is this: if z tends to infinity *while staying a fixed finite distance to the right of the imaginary axis*, then $J(z)$ tends to 0. In other words,

$$\lim_{z\to\infty,\mathrm{Re}(z)\geq c} J(z) = 0$$

where c is any fixed positive real number.

Incidentally, the limit relation $J(z) \to 0$ can be replaced by sharper estimates. Ahlfors just hints at this. For example, $|J(n)| < \frac{1}{12(n-1)}$. [Or is it $|J(x)| < \frac{1}{12x}$ for real x?]

Ahlfors uses Stirling's formula to derive the integral form for $\Gamma(z)$:

$$\Gamma(z) = \int_0^\infty e^{-t}t^z\frac{dt}{t}, \quad \text{for } \mathrm{Re}(z) > 0$$

The factor $\sqrt{2\pi}$ isn't needed for his argument, but the validity of Stirling's formula for complex z is.

Branches of the Logarithm: The derivation makes extensive use of logarithms of complex arguments, so we have to worry about branches of $\log z$. First note that $\log z$ is well-defined modulo $2\pi i$, so, e.g.:

$$\log AB \equiv \log A + \log B \pmod{2\pi i}$$

It's easiest to interpret some of Ahlfors's equations first as congruences modulo $2\pi i$, and only later worry about branches.

OK, so let's worry. $\Gamma(z)$ is a single-valued function, but Stirling's approximation uses both $\sqrt{z}$ and z^z, which are multiple-valued. In other words, Stirling's formula won't survive analytic continuation around zero. However, we want to restrict z to the right half plane anyway. So we can specify the principal branch of $\log z$, $\sqrt{z}$, and z^z.

So much for the right hand side. Now what about $\log \Gamma(z)$, do we want the principal branch of the logarithm there? We will see in a moment that we *don't*. But if z is real and positive, then so is $\Gamma(z)$ (just look at the product representation for $\Gamma(z)$). So no problem in this case: we take real logs on both sides.

For complex z in the right half plane, we just draw a path in the half plane connecting z to some point on the positive real axis, and analytically continue $\log \Gamma(\zeta)$ from that point to z. Since $\Gamma(\zeta)$ is neither zero nor singular in the right half plane, this is possible. (In fact, $\Gamma(\zeta)$ is never zero *anywhere*.) Also, since the right half plane is simply connected, this definition for $\log \Gamma(z)$ doesn't depend on the choice of path (or the initial positive real).

Now for some gory details, not logically necessary, but illuminating.

Say $z = re^{i\theta}$. Then $\log z = \log r + i\theta$, where θ is strictly between $-\pi$ and π if we specify the principal branch. Expanding out the right hand side of Stirling's approximation, we get

$$\operatorname{Re}(\log \Gamma(z)) \approx r \log r \cos\theta - r\theta \sin\theta - r\cos\theta - \frac{\log r}{2} + \frac{\log 2\pi}{2}$$

and

$$\mathrm{Im}(\log\Gamma(z)) \approx r\log r\sin\theta + r\theta\cos\theta - r\sin\theta - \frac{\theta}{2}$$

Let's pick a fixed non-zero value for θ in the interval $(-\pi/2, \pi/2)$. Now let $r \to +\infty$. In the expression for $\mathrm{Im}(\log\Gamma(z))$, the term $r\log r\sin\theta$ dominates, so the imaginary part of $\log\Gamma(z)$ becomes arbitrarily large, and so $\Gamma(z)$ spirals around the orgin an infinite number of times as z tends to infinity along the ray $\theta = \theta_0 \neq 0$.

Although $\Gamma(z)$ is single-valued in the entire plane, its logarithm becomes multiple-valued if we continue it around the poles of $\Gamma(z)$. But of course any two values for $\log\Gamma(z)$ differ by a multiple of $2\pi i$. This is *not* true for Stirling's approximation, as can be seen from the expressions above. Applying exp to $\log\Gamma(z)$ wipes out all ambiguity; not so for the approximation.

2 Plan of Attack

Details in the next section.

Step 1:

$$\frac{d^2}{dz^2}\log\Gamma(z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$$

Step 2:

$$\sum_{n=0}^{\infty}\frac{1}{(z+n)^2} = \frac{1}{2z^2} - \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\pi\cot(\pi\zeta)}{(z+\zeta)^2}d\zeta$$

by contour integration. Assume $\mathrm{Re}(z) > 0$.

Step 3:

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\pi\cot(\pi\zeta)}{(z+\zeta)^2}d\zeta &= \frac{1}{2}\int_0^\infty \cot(\pi i\eta)\left(\frac{1}{(z+i\eta)^2}-\frac{1}{(z-i\eta)^2}\right)d\eta \\ &= -\int_0^\infty \coth(\pi\eta)\frac{2\eta z}{(\eta^2+z^2)^2}d\eta\end{aligned}$$

by simple manipulations: split the integral in two parts, and use substitution.

Step 4: Put steps 1–3 together, and do a little more algebra:

$$\frac{d^2}{dz^2}\log\Gamma(z)=\frac{1}{z}+\frac{1}{2z^2}+\int_0^\infty\frac{4\eta z}{(\eta^2+z^2)^2}\frac{d\eta}{e^{2\pi\eta}-1}$$

Step 5: Integrate step 4 with respect to z:

$$\frac{d}{dz}\log\Gamma(z)=C+\log z-\frac{1}{2z}-\int_0^\infty\frac{2\eta}{\eta^2+z^2}\frac{d\eta}{e^{2\pi\eta}-1}$$

Step 6: Integrate by parts:

$$\int_0^\infty\frac{2\eta}{\eta^2+z^2}\frac{d\eta}{e^{2\pi\eta}-1}=-\frac{1}{\pi}\int_0^\infty\frac{z^2-\eta^2}{(\eta^2+z^2)^2}\log(1-e^{-2\pi\eta})d\eta$$

Step 7: Integrate again with respect to z, replacing $C-1$ by C (after all, it's only a constant of integration):

$$\log\Gamma(z)=C'+Cz+\left(z-\frac{1}{2}\right)\log z+\frac{1}{\pi}\int_0^\infty\frac{z}{\eta^2+z^2}\log\frac{1}{1-e^{-2\pi\eta}}d\eta$$

So now we have Stirling's formula, except for the constants of integration and the limit relations:

$$\log\Gamma(z) = C' + Cz + z\log z - \frac{1}{2}\log z + J(z)$$

with

$$J(z) = \frac{1}{\pi}\int_0^\infty \frac{z}{\eta^2+z^2}\log\frac{1}{1-e^{-2\pi\eta}}d\eta$$

Step 8: $J(z) \to 0$ as $z \to \infty$ with $\mathrm{Re}(z) \geq c$, for any fixed real positive c.

Step 9: $C = -1$. This comes from the functional equation

$$z\Gamma(z) = \Gamma(z+1)$$

or

$$\log z + \log\Gamma(z) = \log\Gamma(z+1)$$

by letting z tend to infinity and using step 8.

Step 10: $C' = \frac{1}{2}\log(2\pi)$. Here we use

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

and again let $z \to \infty$. To keep both z and $1-z$ in a suitable half-plane, let $z = \frac{1}{2} + i\eta$, $1 - z = \frac{1}{2} - i\eta$, $\eta \to +\infty$.

3 Details

Step 1: Why is this a more convenient starting point than the product representation, or its logarithm?—

$$\log\Gamma(z) = -\gamma z - \log z + \sum_{n=1}^{\infty}\left(\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right)$$

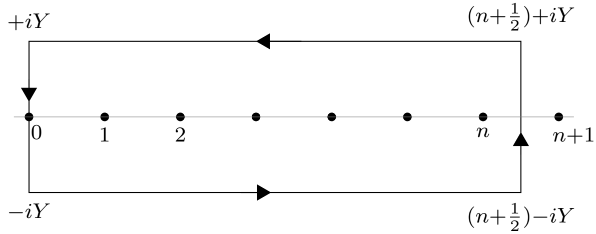

Figure 1: Contour of Integration

Well, first of all we have a nice simple absolutely convergent series $\sum_{n=0}^{\infty} 1/(z+n)^2$ (for $\mathrm{Re}(z) > 0$) instead of a delicate cancellation of two divergent series, $\sum_{n=1}^{\infty} z/n$ and $\sum_{n=1}^{\infty} \log(1+\frac{z}{n})$.

Also, taking the second derivative gets rid of the constant γ. The price of this is steps 9 and 10 to determine the constants of integration, but since the answers don't resemble γ at all, we haven't lost anything.

Step 2: Let's write

$$\Phi_z(\zeta) = \frac{\pi \cot(\pi\zeta)}{(z+\zeta)^2}$$

to emphasize the dependence on z. Now, $\pi \cot(\pi\zeta)$ has residue 1 at integer n (see p. 187 eqn. 10 §2.1), so $\Phi_z(\zeta)$ has residue $\frac{1}{(z+n)^2}$ at integer n, *provided* z is not $-n$; otherwise we have to worry about what happens when the double pole at $-z$ sits on top of the simple pole from the cotangent. Since we are interested only in the residues at the non-negative integers, and since we assume that the real part of z is positive, we are spared these complications.

The contour we use is shown in figure 1.

(Note the minus sign in front of the integral in the step 1 formula, because this contour goes from $+i\infty$ to $-i\infty$ on the imaginary axis, but the integral goes in the contrary direction. This is what Ahlfors means when he says later "The sign has to be reversed…".)

As Ahlfors notes, this contour cuts through the pole at 0, giving rise to the term $\frac{1}{2z^2}$, if we use the Cauchy principal value for the integral.

First we send Y to infinity. We need the limit of $\cot(\pi(\xi \pm iY)$ as $Y \to \infty$. Rewrite $\cot Z$ in this form:

$$\begin{aligned}\cot Z &= i\frac{e^{iZ}+e^{-iZ}}{e^{iZ}-e^{-iZ}} \\ &= i\frac{e^{2iZ}+1}{e^{2iZ}-1} \\ &= i\frac{1+e^{-2iZ}}{1-e^{-2iZ}}\end{aligned}$$

If $Z = A + iB$, then $|e^{2iZ}| = e^{-2B}$, so as $B \to +\infty$, $\cot Z \to -i$. Likewise, $|e^{-2iZ}| = e^{2B}$, so $\cot Z \to i$ as $B \to -\infty$.

Ahlfors evaluates the integral over the right-hand vertical line

$$\int_{\xi=n+(1/2)} \frac{d\eta}{|\zeta+z|^2}$$

using residues. This is cute, but perhaps overkill. For n large enough, ζ swamps z, and $|\zeta+z|^2$ is essentially $n^2+\eta^2$, so the integral is bounded by a constant times $\int_{-\infty}^{+\infty} \frac{d\eta}{n^2+\eta^2}$, i.e., $\frac{1}{n}\arctan\eta\big|_{-\infty}^{+\infty}$, which tends to 0 as n tends to infinity. (To make this rigorous, pick n so large that $|\zeta| > 2|z|$ all along the vertical axis. Then $\frac{1}{|\zeta+z|^2} < \frac{4}{|\zeta|^2}$.)

Step 3: Ahlfors has a misprint, writing $\frac{1}{(i\eta+z^2)}$ instead of $\frac{1}{(i\eta+z)^2}$.

To obtain

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\pi\cot(\pi\zeta)}{(z+\zeta)^2}d\zeta = \frac{1}{2}\int_0^\infty \cot(\pi i\eta)\left(\frac{1}{(z+i\eta)^2} - \frac{1}{(z-i\eta)^2}\right)d\eta$$

split the integral into $\int_{-i\infty}^{0} + \int_{0}^{+i\infty}$. Then substitute $i\eta = \zeta$ in $\int_{0}^{+i\infty}$, and $-i\eta = \zeta$ in $\int_{-i\infty}^{0}$. The i from $d\zeta = d(\pm i\eta)$ cancels the i in the factor of $\frac{1}{2\pi i}$ out front. We get three minus signs for the integral $\int_{-i\infty}^{0}$: one from $d\zeta = d(-i\eta)$, one from switching the limits of integration, and one because the cotangent is an odd function.

The rest of this step is simple algebra, plus the identity $\cot(iZ) = -i\coth(Z)$.

But what's the advantage to doing this? Well, note that when z is much larger than η, $\frac{1}{(z+i\eta)^2}$ and $\frac{1}{(z-i\eta)^2}$ are nearly the same, so subtracting them we get a smaller integrand than before. And the same is true if η is much larger than z. At the hand-waving level, having a small integrand is a Good Thing when you're trying to estimate an integral. We'll see this same philosophy at work in the next step.

Step 4: The trick of writing

$$\coth \pi\eta = 1 + \frac{2}{e^{2\pi\eta-1}}$$

is undoubtedly motivated by the observation that $\lim_{X\to+\infty} \coth X = 1$. The verification starts off as the hyperbolic counterpart to what we did with the cotangent above:

$$\begin{aligned}
\coth X &= \frac{e^X + e^{-X}}{e^X - e^{-X}} \\
&= \frac{e^{2X}+1}{e^{2X}-1} \\
&= \frac{e^{2X}-1+2}{e^{2X}-1} \\
&= 1 + \frac{2}{e^{2X}-1}
\end{aligned}$$

We need to integrate $\int_0^\infty \frac{2\eta z}{(\eta^2+z^2)^2} d\eta$. This yields immediately to the substitution $v = \eta^2 + z^2$:

$$\int \frac{2\eta z}{(\eta^2+z^2)^2} d\eta = \int \frac{z\,dv}{v^2} = \frac{-z}{\eta^2+z^2}$$

$$\int_0^\infty \frac{2\eta z}{(\eta^2+z^2)^2} d\eta = 0 - \frac{-z}{z^2} = \frac{1}{z}$$

So the remaining integral is "very strongly convergent" because we've separated out the asymptotic limit of $\coth 2\pi\eta$ (namely 1) and integrated that part exactly.

We can now see another benefit flowing from step 3. $\lim_{X\to+\infty} \coth X = 1$, but $\lim_{X\to-\infty} \coth X = -1$. If we hadn't converted the limits of integration from $\int_{-\infty}^{+\infty}$ to $\int_0^\infty$, we couldn't pull out the asymptotic part so easily.

Step 5: The integral on the right hand side looks more fearsome than it is. Since we're integrating with respect to z, we basically have to evaluate $\int \frac{z\,dz}{(\text{constant } + z^2)^2}$, which is easy.

Branch questions: basically, we've shown that $f'(z) = g'(z)$, where f and g are analytic functions in the right half plane. So there is some constant C that makes the equation for step 5 true. So far we could use any analytic branch for $\log z$ and $\log \Gamma(z)$. Actually, the choice of branch for $\log \Gamma(z)$ doesn't matter yet.

Step 6: Ahlfors has a misprint: he omits the minus sign on the right hand side.

The integration by parts goes as follows (remembering to treat z as a constant):

$$u = \tfrac{2\eta}{\eta^2+z^2} \qquad du = \tfrac{2(z^2-\eta^2)}{(\eta^2+z^2)^2} d\eta$$

$$dv = \tfrac{d\eta}{e^{2\pi\eta}-1} \quad v = \tfrac{1}{2\pi}\log(1-e^{-2\pi\eta})$$

so

$$\int_0^\infty \frac{2\eta}{\eta^2+z^2}\frac{d\eta}{e^{2\pi\eta}-1} = \frac{1}{\pi}\frac{\eta}{\eta^2+z^2}\log(1-e^{-2\pi\eta})\Big|_0^\infty - \frac{1}{\pi}\int_0^\infty \frac{z^2-\eta^2}{(\eta^2+z^2)^2}\log(1-e^{-2\pi\eta})d\eta$$

We have to show that the integrated part vanishes. This is obvious for the contribution from $\eta = \infty$. For $\eta = 0$ however we get the indeterminate form $0 \cdot \log 0$. We can either use l'Hôpital's rule, or argue that for η close to zero, we basically have $\eta \log 2\pi\eta$, and $\lim_{\eta\to 0} \eta \log \eta = 0$ (e.g., look at $\eta = e^{-100}$).

Step 7: Note that $\int \log z\, dz = z \log z - z +$ constant. We absorb the $-z$ into the constant C, as noted.

Branch questions are slightly trickier this time, but still not bad. Now we've shown basically that $f''(z) = g''(z)$ for analytic functions f and g in the right half plane. So there are constants C and C' so that the equation for step 7 is true.

For simplicity, we choose the principal branch of $\log z$, the branch of $\log \Gamma(z)$ specified before, and the real branch of $\log \frac{1}{1-e^{-2\pi\eta}}$. This is what Ahlfors means when he says, "By proper choice of C' we obtain the branch of $\log \Gamma(z)$ which is real for real z."

It follows, incidentally, that the integral on the right hand side must change by a constant, independent of z, if we choose a different branch for $\log \frac{1}{1-e^{-2\pi\eta}}$. This is easily verified.

Step 8: First off, we show that $\int_0^\infty \log(1-e^{-2\pi\eta})d\eta$ is convergent. Substitute

$$u = e^{-2\pi\eta} \quad du = -2\pi e^{-2\pi\eta}d\eta = -2\pi u d\eta$$

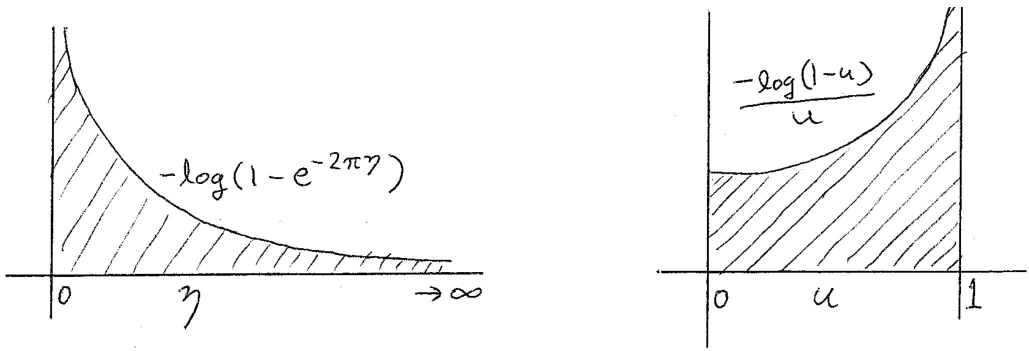

Figure 2: Integrals

Here's a crib sheet to keep the limits of integration straight.

η	$u = e^{-2\pi\eta}$	$1-u$	$-\log(1-u)$	$-\log(1-u)/u$
∞	0	1	0	1
0	1	0	$+\infty$	$+\infty$

So

$$\int_0^\infty -\log(1-e^{-2\pi\eta})d\eta = \frac{1}{2\pi}\int_1^0 \frac{\log(1-u)du}{u} = \frac{1}{2\pi}\int_0^1 \frac{-\log(1-u)du}{u}$$

So the integrals look like figure 2.

The second integral is known as the dilogarithm:

$$\text{dilog}(\rho) = \int_0^\rho \frac{-\log(1-u)du}{u} = \rho + \frac{\rho^2}{4} + \ldots + \frac{\rho^n}{n^2} + \ldots$$

as we get from integrating the Taylor series for $\log(1-u)$ term by term. At $\rho = 1$ we get the convergent series $\sum_{n=1}^\infty 1/n^2$. So the convergence of this integral is not entirely trivial. Lang's *Complex Analysis* (3rd edition) has more to say about the dilogarithm, in particular its analytic continuation.

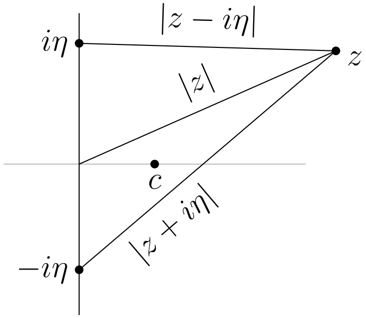

Figure 3: Why both factors are $> c$

Note that we use here, for the first time, one of our branch assumptions. The Taylor series for $\log(1 - u)$ picks out the principal branch of the logarithm in a neighborhood of 1.

From the convergence of $\int_0^\infty \log(1 - e^{-2\pi\eta})d\eta$ Ahlfors concludes that

$$-\int_0^\infty \frac{z}{\eta^2 + z^2} \log(1 - e^{-2\pi\eta})d\eta$$

tends to zero as $z \to \infty$ with $\mathrm{Re}(z) > c$. Note the general strategy: when η is small compared to $|z|$, the fraction is approximately $1/z$, so that part is under control. When η is about the same size as z or greater, at first it seems that the fraction is less than $1/\eta$. Things aren't that simple, though: just try setting $z = i\eta$. This is why we have to keep z away from the imaginary axis.

The key trick: $|\eta^2 + z^2| = |z - i\eta||z + i\eta|$. The following diagram (figure 3) shows why both factors must be $> c$, and at least one factor must be $\geq |z|$. Formally, if both factors were $< |z|$, then we would have $|2z| = |(z+i\eta)+(z-i\eta)| \leq |z+i\eta|+|z-i\eta| < 2|z|$. Also, $\mathrm{Re}(z \pm i\eta) = \mathrm{Re}(z) > c$, and for any complex number a, $|a| \geq |\mathrm{Re}(a)|$.

So the first part of the integral (Ahlfors' J_1) is small because the integrand in question is small, thanks to efforts to make it small in steps 3 and 4. The second part of the integral (J_2) is small because it's bounded by a constant times the tail of the convergent dilogarithmic integral.

Step 9: This part is pretty straightforward. However, branch aspects are worth closer scrutiny. From $z\Gamma(z) = \Gamma(z+1)$ we conclude at first only $\log z + \log\Gamma(z) \equiv \log\Gamma(z+1) \pmod{2\pi i}$. But we can replace $\equiv$ with $=$ on the positive real axis, where by assumption we use the real branch of log both for $\log z$ and $\log\Gamma(z)$. We take the limit as $z \to \infty$ through positive real values, so that settles that. Note also that the limit $\lim_{z\to\infty} z\log\left(1+\frac{1}{z}\right) = 1$ requires the principal branch.

Step 10: The factor $\sqrt{2\pi}$. According to Graham, Knuth, and Patashnik, *Concrete Mathematics*, it took Stirling several years to find this last piece of his formula.

Starting from $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, we get first:

$$\log\Gamma(z) + \log\Gamma(1-z) \equiv \log\pi - \log\sin\pi z \pmod{2\pi i}$$

Now for $z = \frac{1}{2}$, all the arguments are real, so by our branch conventions we can replace the congruence with an equality. By analytic continuation over the right half plane, the equation holds for all z with $\mathrm{Re}(z) > 0$.

This time we let z tend to infinity upwards along the line $z = \frac{1}{2} + iy$, for as noted earlier, this keeps $1 - z = \frac{1}{2} - iy$ in the right half plane also. Step 9 used the relation $\lim_{z\to\infty} J(z) = 0$ only as z tended to infinity along the real axis; this weaker result could have been obtained with less effort.

The rest of the argument is straightforward, with only a few points to note. First, $\sin(\frac{\pi}{2} + \pi iy) = \cos(\pi iy) = \cosh(\pi y)$. Next, $\log(\frac{1}{2}+iy) - \log(\frac{1}{2}-iy) = \log\left(\frac{\frac{1}{2}+iy}{\frac{1}{2}+iy}\right)$ because the angle between $\frac{1}{2}+iy$ and $\frac{1}{2}-iy$ is between 0 and π, so we remain inside the domain of the principal branch. For the same reason,

$$\log\left(-\frac{\frac{1}{2}+iy}{\frac{1}{2}+iy}\right) = -i\pi + \log\left(\frac{\frac{1}{2}+iy}{\frac{1}{2}+iy}\right)$$

which Ahlfors uses at the bottom of page 202. To get the Taylor expansion for $\log\frac{1+t}{1-t}$, just rewrite it as $\log(1+t) - \log(1-t)$, valid in a neighborhood of 1 if we use principal logs. So we have

$$\log\frac{1+t}{1-t} = t + \frac{t^3}{3} + \frac{t^5}{5} + \dots$$

and so

$$\log\frac{1+\frac{1}{2iy}}{1-\frac{1}{2iy}} = 2\left(\frac{1}{2iy} + \frac{1}{3(2iy)^3} + \dots\right)$$

Also,

$$\log\cosh\pi y = \log\frac{e^{\pi y}+e^{-\pi y}}{2} = \log\frac{1+e^{-2\pi y}}{2e^{-\pi y}} = -\log 2 + \pi y + \log(1+e^{-2\pi y})$$

Putting it all together, we have:

$$2C' - 1 - \pi y + 1 + \epsilon_1(y) + \epsilon_J(y) = \log\pi - \pi y + \log 2 - \epsilon_2(y)$$

(where $\epsilon_J(y)$ comes from $J(\frac{1}{2}+iy) + J(\frac{1}{2}-iy)$, of course). It's no accident that the πy's cancel, of course— since C' is constant, any terms with y in them have to cancel. Probably one could use this fact to be somewhat sloppier about branches.

So $C' = \frac{1}{2}\log 2\pi$, and we are done.

4 Other Treatments

The most popular derivation seems to be by way of Euler-Maclaurin summation. Graham, Knuth, and Patashnik give a detailed treatment in the last section of *Concrete Mathematics*. Spivak's *Calculus* develops it in an exercise (Problem 25–18, first edition). Lang gives a slightly simplified (or disguised) presentation in his *Complex Analysis*, and a heavily disguised one in his *Undergraduate Analysis* (the analysis of undergraduates?) He does apologize there for pulling rabbits out of hats.

Bamberg and Sternberg, *A Course in Mathematics for Students of Physics*, volume II, give a derivation based on Laplace's method, and relate this to the method of stationary phase.

Finally, Walsh's notes on Potential Theory give a derivation based (I think) on the Bohr-Mollerup theorem.