Stirling's Formula

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Stirling's formula is

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

or in logarithmic form

$$\log n! \approx n \log n - n + \frac{1}{2}\log n + \frac{1}{2}\log 2\pi$$

One has to work moderately hard to obtain this result. To get a better feel for what's going on, I looked at some cruder estimates that can be obtained without breaking a sweat.

First approximation:

$$n! \approx n^n, \qquad \log n! \approx n \log n$$

This is such a crude approximation that you'll probably feel more comfortable if I write it as an inequality:

$$n! < n^n, \qquad \log n! < n \log n$$

Second approximation:

$$n! \approx \left(\frac{n}{e}\right)^n, \qquad \log n! \approx n \log n - n$$

This comes from the series for e^n:

$$e^n = 1 + n + \ldots + \frac{n^n}{n!} + \ldots \approx \frac{n^n}{n!}$$

so

$$n! \approx \left(\frac{n}{e}\right)^n$$

This time we get the inequality

$$n! > \left(\frac{n}{e}\right)^n, \qquad \log n! > n \log n - n$$

So combined with the first inequality, we have determined $n!$ up to a factor of e^n. This may seem ridiculously crude, but it does mean that we've determined that $(1/n) \sum_{k=1}^n \log k$ is between $\log n$ and $(\log n) - 1$. For some purposes, this is good enough.

Third approximation:

$$n! \approx \left(\frac{n}{e}\right)^n e, \qquad \log n! \approx n \log n - n + 1$$

This comes from the observation that $\log n! = \sum_{k=1}^n \log k$, plus the old trick of approximating a sum by an integral (see figure 1).

We have (freely using the fact that $\log 1 = 0$):

$$\int_1^n \log x \, dx < \sum_{k=1}^n \log k < \int_2^{n+1} \log x \, dx$$

or

$$n \log n - n + 1 < \log n! < (n+1) \log (n+1) - (n+1) - 2 \log 2 + 2$$

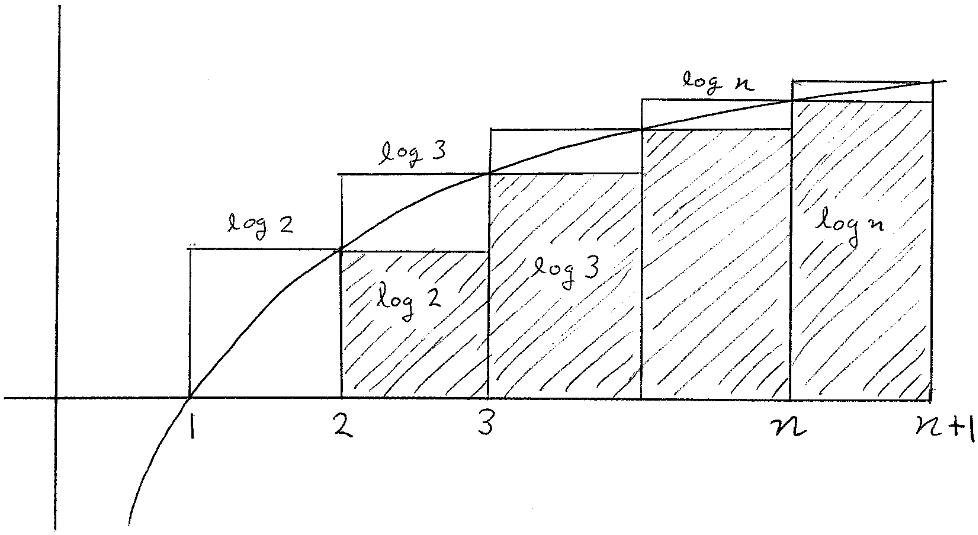

Figure 1: Riemann Sums

or rewriting to make the lower and upper bounds as similar as possible:

$$n \log n - n + 1 < \log n! < n \log (n+1) - n + \log (n+1) - (\log 4 - 1)$$

and in exponential form:

$$\left(\frac{n}{e}\right)^n e < n! < \left(\frac{n+1}{e}\right)^n (n+1)\frac{e}{4}$$

Now the ratio of $\left(\frac{n+1}{e}\right)^n$ to $\left(\frac{n}{e}\right)^n$ is just $\left(\frac{n+1}{n}\right)^n$, which tends to e as n tends to infinity. So ignoring constant factors, we can say that the important difference between the lower and upper bounds is that factor of $n+1$ on the right. Loosely speaking, we've determined $n!$ up to a factor of n.

Fourth approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{n}e, \qquad \log n! \approx n \log n - n + \frac{1}{2} \log n + 1$$

In the third approximation, we compared an integral with some Riemann sums. So the next obvious idea is to use a better method of approximating integrals (see figure 2).

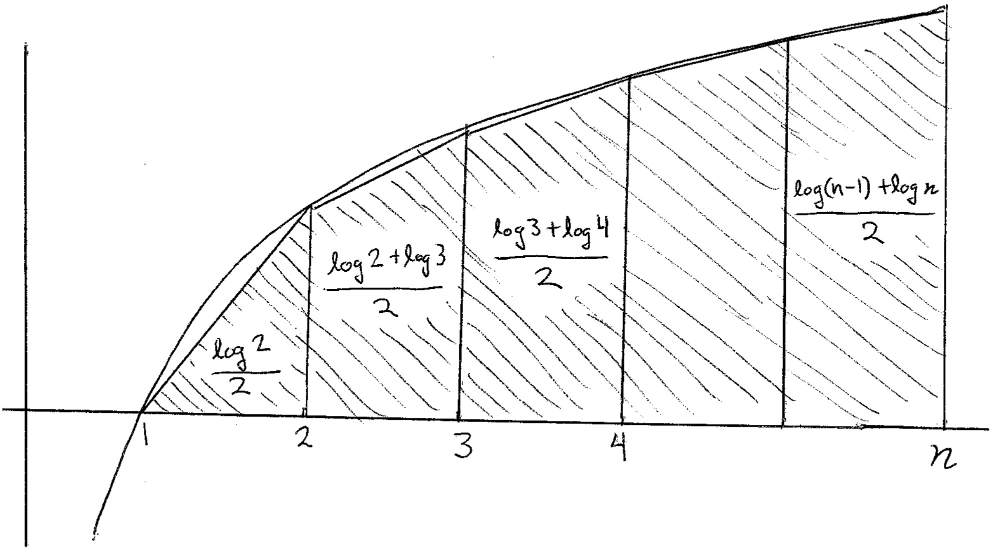

Figure 2: Trapezoidal Rule

The trapezoidal rule says that:

$$\int_{x_1}^{x_n} f(x)dx \approx \frac{\Delta x}{2}(f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n))$$

We apply this to $\int_1^n \log x\, dx$. In fact it's clear that the trapezoidal approximation is less than the integral in this case, so

$$\frac{1}{2}(\log 1 + 2\log 2 + \ldots + 2\log(n-1) + \log n) < n\log n - n + 1$$

or adding $(\log n)/2$ to each side,

$$\log n! < n\log n - n + \frac{1}{2}\log n + 1, \qquad n! < \left(\frac{n}{e}\right)^n \sqrt{n}e$$

As an approximation, this is the same as Stirling's formula except for replacing $\sqrt{2\pi}$ with e. Pulling out ye olde calculator, we find:

$$\sqrt{2\pi} = 2.5\ldots, \quad e = 2.7\ldots$$

Not bad!

Fifth approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{n}\frac{e}{\sqrt{2}}, \qquad \log n! \approx n\log n - n + \frac{1}{2}\log n + 1 - \frac{1}{2}\log 2$$

Hey, what gives— this isn't even as good as the last approximation. (Our trusty calculator tells us that $e/\sqrt{2}$ is about 1.9.)

Well, the real point are the inequalities. We have an upper bound; how to get a lower bound? Not too hard: we cover the area defined by the integral $\int_1^n \log x\,dx$ with a collection of trapezoids (see figure 3).

This time, we choose a different line segment for the upper edge of each trapezoid. Namely, for the trapezoid from $x = k-1$ to $x = k$, we draw the line tangent to $\log x$ at $(k, \log k)$ and extend it left to the vertical line $x = k-1$. The slope of the tangent line is $1/k$, so the height of left vertical edge of the trapezoid is $\log k - 1/k$, and the trapezoid's area is $(1/2)(\log k + \log k - 1/k)$. So we have:

$$\int_1^n \log x\,dx < (\log 2 + \ldots + \log n) - \frac{1}{2}\left(\frac{1}{2} + \ldots + \frac{1}{n}\right)$$

So:

$$n\log n - n + 1 + \frac{1}{2}\left(\frac{1}{2} + \ldots + \frac{1}{n}\right) < \log n!$$

Now we need a lower bound for $\frac{1}{2} + \ldots + \frac{1}{n}$. We use the same trick of comparing the sum with an integral, namely $\int dx/x$. Result:

$$\log(n+1) - \log 2 < \frac{1}{2} + \ldots + \frac{1}{n} < \log n$$

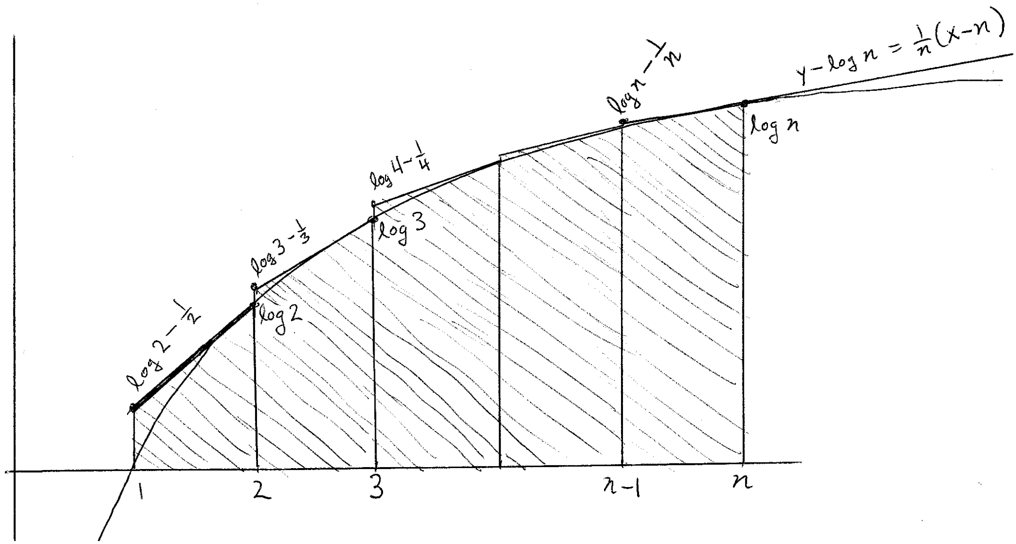

Figure 3: Trapezoidal Lower Bound

We need only the lower bound. In fact, I'm going to sacrifice a small amount of precision for a neater formula, and replace $\log(n+1)$ with $\log n$. (After all, it still wouldn't be as good as Stirling's formula.)

So our final conclusion:

$$n \log n - n + \frac{1}{2} \log n + 1 - \frac{1}{2} \log 2 < \log n! < n \log n - n + \frac{1}{2} \log n + 1$$

$$\left(\frac{n}{e}\right)^n \sqrt{n} \frac{e}{\sqrt{2}} < n! < \left(\frac{n}{e}\right)^n \sqrt{n} e$$

So we've determined $n!$ to within a factor of $\sqrt{2}$.

However, Stirling's result is really more than just the approximation. The two sides are *asymptotic*:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

i.e., the ratio of the two sides tends to 1 as n tends to infinity.

So the real heavy lifting for Stirling's formula comes up when you try to prove two things: (a) the ratio of $n!$ to $(n/e)^n\sqrt{n}$ does approach a limit, and (b) this limit is $\sqrt{2\pi}$.

By way of comparison, Chebyshev showed in 1852 that

$$Kn/\log n < \pi(n) < Ln/\log n$$

for some constants K and L (where $\pi(n)$ is of course the number of primes less than or equal to n). He also showed that if $\frac{\pi(n)}{n/(\log n)}$ does approach a limit, this limit must be 1. He used only elementary methods. But it wasn't until 1896 that Hadamard and de la Valleée Poussin proved the Prime Number Theorem, using real power tools.