# Models of True Arithmetic that are not Omegas

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By an $\omega$, we mean $\omega^V$ where $V \models$ ZF.

**Theorem:** Any nonstandard model of PA has an elementary submodel which is not an $\omega$.

**Proof:** Let $M$ be the model. There is a countable collection $\mathcal{F}$ of Skolem functions such that if $A \subseteq M$ is closed under $\mathcal{F}$, then $A \preceq M$. Let $\{f_n | n \in \mathbb{N}\}$ be an enumeration of the compositions of functions in $\mathcal{F}$, so that

$$N_a = \{f_n(a) | n \in \mathbb{N}\}$$

is the closure of $\{a\}$ under $\mathcal{F}$. If $a$ is nonstandard, then $N = N_a$ will be the desired submodel.

To show that $N$ is not an $\omega$, we first make three observations:

1. For any $n \in \mathbb{N}$, the relation $f_n(x) = y$ (in $x$ and $y$) is definable in PA, say by a formula
$$\varphi_n(x, y) \equiv f_n(x) = y$$

2. The relation $f_n(x) = y$ in $n$, $x$, and $y$, is definable in ZF, say by a formula
$$\Phi(n, x, y) \equiv f_n(x) = y$$

3. For all $n \in \mathbb{N}$,

$$\mathrm{ZF} \vdash \forall x, y[\varphi_n(x, y) \leftrightarrow \Phi(n, x, y)]$$

By (1), we can define $f_n(x)$ inside $N$ for all $n \in \mathbb{N}$. Since $N \preceq M$, this agrees with the $f_n$ obtained from $M$.

By (2), we can define $F(n, x)$ in $V$ for any $V \models \mathrm{ZF}$. ($F$ will be a function $\omega^V \times \omega^V \to \omega^V$.)

By (3), if $N = \omega^V$ then for all $n \in \mathbb{N}$, $V \models (\forall x \in \omega) f_n(x) = F(n, x)$. So $F(n, \cdot) \equiv f_n(\cdot)$.

Now suppose $N = \omega^V$ for some $V \models \mathrm{ZF}$. Then

$$N = \{f_n(a) | n \in \mathbb{N}\} \subseteq \{F(n, a) | n < a\} = N$$

so $\{F(n, a) | n < a\} = N = \omega^V$. But $\{F(n, a) | n < a\}$ is definable in $V$, and in fact

$$V \models [\{F(n, a) | n < a\} \text{ is finite}]$$

This contradiction proves the theorem.