Epstein Relativity Diagrams

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Lewis Carroll Epstein wrote a book *Relativity Visualized*. It's been called "the gold nugget of relativity books". I wouldn't go quite *that* far, but Epstein has devised a completely new way to explain relativity. The key concept: the Epstein diagram. (I should mention that *Relativity Visualized* is a pop-sci treatment.)

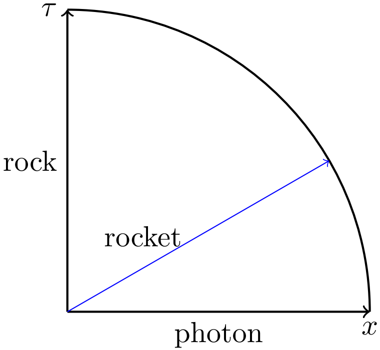
Here's how Epstein introduces his idea:

Why can't you travel faster than light?

> THE REASON YOU CAN'T GO FASTER THAN THE SPEED OF LIGHT IS THAT YOU CAN'T GO SLOWER. THERE IS ONLY ONE SPEED. EVERYTHING, INCLUDING YOU, IS ALWAYS MOVING AT THE SPEED OF LIGHT. *How can you be moving if you are at rest in a chair? You are moving through time.*

This is both poetic and (I feel) unnecessarily mystifying. Let me quickly add: Epstein does not want you to take this literally. It appears in a chapter titled "The Myth", where Epstein advocates for "myths"—I'd call them analogies—to help us grasp the basic concepts of physics. You can't argue with that. (Well you *can*, but you shouldn't.)

I'll undertake demystification in a moment. But first, here's the prototypical Epstein diagram:



As with Minkowski diagrams, we ask two spatial dimensions to leave the room so we can draw on a 2D screen. Note that the vertical axis is labeled τ, for proper time. In a Minkowski diagram, this would be labeled t, of course.

The rock, the rocket, and the photon all enjoy a mythical speed of 1 (setting $c = 1$, as usual). Our rock sits still (in the lab frame, let's say). So its mythical speed consists entirely of motion through time. The photon travels through space as fast as possible, with nothing is left over for motion through time. Its elapsed proper time is zero. The rocket devotes some mythical speed to spatial motion, some to temporal motion. I've illustrated the case where $dx/dt = \sqrt{3}/2$, so in one coordinate second it covers approximately 0.87 light-seconds of distance and 1/2 second of proper time.

OK, what's really going on? Start with the Minkowski metric

$$d\tau^2 = dt^2 - dx^2$$

and rewrite it

$$dt^2 = d\tau^2 + dx^2.$$

Hey, that looks just like the Pythagorean formula! Can we banish the strange non-euclidean features of Minkowski space, and revert to good old high-school geometry?

Of course not. There's a fly in the ointment. Unlike dt and dx, $d\tau$ is not an exact differential, so you can't use (τ, x) as a coordinate system for spacetime. Not if you want a 1–1 correspondence between events and coordinate pairs. You *can* use τ as a coordinate along *one worldline*, however (provided the particle keeps below the speed of light).

Think of Epstein diagrams as a graphical calculation technique. Say we have the worldline of a particle

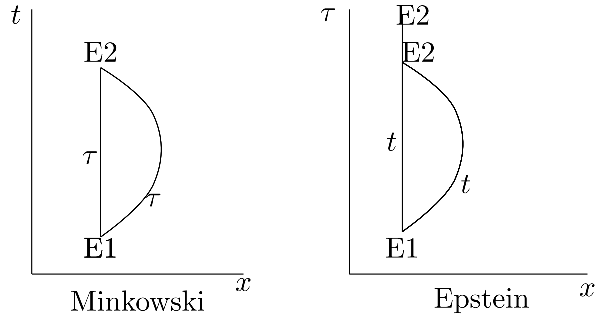
$$x = f(t).$$

So long as the particle obeys the speed limit, τ is a strictly increasing function of t. That means you can also give the particle's position as a function of its proper time:

$$x = g(\tau).$$

The graph of g is the Epstein diagram. Since $dt^2 = d\tau^2 + dx^2$, the (ordinary euclidean) length of the curve is the elapsed *coordinate* time.

For example, look at this diagram, illustrating the twin paradox:



With our familiar Minkowski diagram, we have points E1 and E2 for the events "twins bid each other farewell" and "twins joyfully reunite". The elapsed time for the stay-at-home twin Terrence is *longer* than for his adventurous sister Stella, even though it's *shorter* in the diagram.

In the Epstein diagram, we draw curves for g_{Terrence} and g_{Stella}, starting at the same point. Both curves have the same length, representing the coordinate time between E1 and E2. As you can clearly see, Stella ages less between E1 and E2 than Terrence.

Epstein uses a variation to explain length contraction. Once again, he obscures what is really going on by expressing the matter "mythically". Let's consider a station-train scenario, standard ever since Einstein. A train whizzes past a station. The stationmaster wants to measure the length of the train. This amounts to measuring the distance between two events, one at the front of the train (call it E for Engine), and one at the rear (C for Caboose). C and E must be simultaneous for the stationmaster. C and E won't be simultanous for the passengers on the train, but they don't care: in their reference frame, the train is not moving, so the distance between *any* two events, one at the front and one at the rear, equals the length of the train.

C and E have a spacelike separation. For such event pairs, the Minkowski metric is written like so:

$$ds^2 = dx^2 - dt^2$$

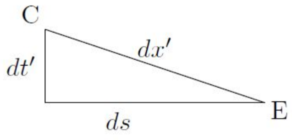
The interval ds is a relativistic invariant, so using passenger coordinates:

$$ds^2 = dx'^2 - dt'^2$$

Rewrite this as

$$ds^2 + dt'^2 = dx'^2$$

and illustrate it with this diagram:



$$ds^2 + dt'^2 = dx'^2$$

Now, dx' is the passengers' measurement of the train-length, while ds is the stationmaster's. (Since C and E are simultaneous in his frame, he has $dt = 0$ and $ds = dx$.) Sure enough, ds *looks* shorter than dx'; what you expect from the word "contraction". In the Minkowski version of this diagram, the hypotenuse is labeled dx and the bottom leg is labeled dx'; the vertical leg is still labeled dt'. Visually dx' looks *shorter* than dx, but it's actually longer in the Minkowki metric. Our eyes, of course, see a Euclidean world.

(Epstein's illustrations for length contraction offer more pizzazz than mine, but he labels the axes in a misleading manner.)

I fear this demystification might undersell the charms of *Relativity Visualized*. Epstein exploits his diagrams for all they're worth when he gets to General Relativity. I particularly like his explanation of the Shapiro time delay.

Epstein has other arrows in his quiver: a fluid writing style, a gift for analogy, and artistic talent (he drew all the illustrations). I award this book the Silver, if not the Gold.